Feedback stabilization of diagonal infinite-dimensional systems in the presence of delays

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Workshop: Input-to-state stability and control of infinite-dimensional systems

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Delay boundary control of PDEs

Topic: stability and stabilization of PDEs in the presence of a delay in the boundary conditions.
[Nicaise and Valein, 2007], [Nicaise and Pignotti, 2008] [Krstic, 2009], [Nicaise, Valein, and Fridman, 2009] [Fridman, Nicaise, and Valein, 2010], [Prieur and Trélat, 2018].

Objective: boundary stabilization and regulation control of open-loop unstable PDEs in the presence of a long input delay.

Example: reaction-diffusion equation

$$\begin{aligned} y_t &= y_{xx} + cy \\ y(t,0) &= 0, \quad y(t,L) = u(t-D) \\ y(0,x) &= y_0(x) \end{aligned}$$

- [Krstic, 2009] - backstepping design.
- [Prieur and Trélat, 2018] - spectral reduction and predictor feedback.

Boundary control of PDEs in the presence of a state-delay

Topic: stability and stabilization of PDEs in the presence of a state-delay. [Fridman and Orlov, 2009], [Solomon and Fridman, 2015], [Hashimoto and Krstic, 2016], [Kang and Fridman, 2017], [Kang and Fridman, 2018].

Objective: boundary stabilization of open-loop unstable PDEs in the presence of a state-delay delay.

Example: reaction-diffusion equation

$$
\begin{aligned}
&y_t(t,x) = y_{xx}(t,x) + a(x)y(t,x) + by(t-h,x) \\
&y(t,0) = 0, \quad y(t,L) = u(t) \\
&y(0,x) = y_0(x)
\end{aligned}
$$

- [Hashimoto and Krstic, 2016] - backstepping design.
- [Kang and Fridman, 2017] - Dirichlet/Neumann boundary conditions and time-varying delay - backstepping design.

Spectral reduction methods for control of PDEs

Spectral reduction and finite-dimensional feedback:

1. Spectral reduction.
2. Keep a finite number of modes to build a finite-dimensional truncated model capturing the unstable dynamics of the original PDE.
3. Design a controller for the truncated model.
4. Check that the proposed controller successfully stabilizes the original infinite-dimensional systems.

Early occurrences of this control design method: [Russell, 1978], [Coron and Trélat, 2004], [Coron and Trélat, 2006], etc.

Extension to delay boundary control of a reaction-diffusion equation: [Prieur and Trélat, 2018] by using a predictor feedback [Artstein, 1982]

Outline

Abstract boundary control system

$\mathcal{H}$ is a separable Hilbert space on $\mathbb{K}$, which is either $\mathbb{R}$ or $\mathbb{C}$.

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t}(t) &= \mathcal{A}X(t) + p(t), && t \geq 0 \\ \mathcal{B}X(t) &= u(t), && t \geq 0 \\ X(0) &= X_0 \end{aligned}$$

- $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ a linear (unbounded) operator;
- $\mathcal{B} : D(\mathcal{B}) \subset \mathcal{H} \to \mathbb{K}^m$ with $D(\mathcal{A}) \subset D(\mathcal{B})$ a linear boundary operator;
- $p : \mathbb{R}_+ \to \mathcal{H}$ a distributed disturbance;
- $u : \mathbb{R}_+ \to \mathbb{K}^m$ the boundary control.

Abstract boundary control system

$\mathcal{H}$ is a separable Hilbert space on $\mathbb{K}$, which is either $\mathbb{R}$ or $\mathbb{C}$.

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t}(t) &= \mathcal{A}X(t) + p(t), && t \geq 0 \\ \mathcal{B}X(t) &= u(t), && t \geq 0 \\ X(0) &= X_0 \end{aligned}$$

We assume that $(\mathcal{A}, \mathcal{B})$ is a boundary control system [Curtain and Zwart, 1995]:

1. the disturbance-free operator $\mathcal{A}_0$, defined on the domain $D(\mathcal{A}_0) \triangleq D(\mathcal{A}) \cap \ker(\mathcal{B})$ by $\mathcal{A}_0 \triangleq \mathcal{A}|_{D(\mathcal{A}_0)}$, is the generator of a C_0-semigroup S on $\mathcal{H}$;
2. there exists a bounded operator $L \in \mathcal{L}(\mathbb{K}^m, \mathcal{H})$, called a lifting operator, such that $\mathrm{R}(L) \subset D(\mathcal{A})$, $\mathcal{A}L \in \mathcal{L}(\mathbb{K}^m, \mathcal{H})$, and $\mathcal{B}L = I_{\mathbb{K}^m}$.

Assumed diagonal structure for $\mathcal{A}_0$

A1) $\mathcal{A}_0$ is a Riesz-spectral operator, i.e. it has simple eigenvalues λ_n with corresponding eigenvectors $\phi_n \in D(\mathcal{A}_0)$, $n \in \mathbb{N}^*$ that satisfy:

1. $\{\phi_n,\ n \in \mathbb{N}^*\}$ is a Riesz basis:
   1. $\overline{\underset{n\in\mathbb{N}^*}{\mathrm{span}_{\mathbb{K}}}\ \phi_n} = \mathcal{H}$;
   2. there exist constants $m_R, M_R \in \mathbb{R}_+^*$ such that for all $N \in \mathbb{N}^*$ and all $\alpha_1, \ldots, \alpha_N \in \mathbb{K}$,

$$m_R \sum_{n=1}^{N} |\alpha_n|^2 \leq \left\| \sum_{n=1}^{N} \alpha_n \phi_n \right\|_{\mathcal{H}}^2 \leq M_R \sum_{n=1}^{N} |\alpha_n|^2.$$

2. The closure of $\{\lambda_n,\ n \in \mathbb{N}^*\}$ is totally disconnected, i.e. for any distinct $a, b \in \overline{\{\lambda_n,\ n \in \mathbb{N}^*\}}$, $[a, b] \not\subset \overline{\{\lambda_n,\ n \in \mathbb{N}^*\}}$.

A2) There exist $N_0 \in \mathbb{N}^*$ and $\alpha \in \mathbb{R}_+^*$ such that $\mathrm{Re}\,\lambda_n \leq -\alpha$ for all $n \geq N_0 + 1$.

Spectral reduction

Let $\{\psi_n,\ n \in \mathbb{N}^*\}$ be the dual Riesz-basis of $\{\phi_n,\ n \in \mathbb{N}^*\}$, i.e., $\langle \phi_k, \psi_l \rangle_{\mathcal{H}} = \delta_{k,l}$ for all $k, l \geq 1$.

We define $x_n(t) \triangleq \langle X(t), \psi_n \rangle_{\mathcal{H}}$ the coefficients of the projection of $X(t)$ into the Riesz basis $\{\phi_n,\ n \in \mathbb{N}^*\}$.

$$X(t) = \sum_{n \geq 1} x_n(t) \phi_n$$

$$m_R \sum_{n \geq 1} |x_n(t)|^2 \leq \|X(t)\|^2 \leq M_R \sum_{n \geq 1} |x_n(t)|_{\mathcal{H}}^2$$

Dynamics of the coefficients of projection:

$$\dot{x}_n(t) = \lambda_n x_n(t) + \langle (\mathcal{A} - \lambda_n I_{\mathcal{H}}) L u(t), \psi_n \rangle_{\mathcal{H}} + \langle p(t), \psi_n \rangle_{\mathcal{H}}$$

Finite dimensional truncated model

$$\dot{Y}(t) = AY(t) + Bu(t) + P(t),$$

where

$$A = \mathrm{diag}(\lambda_1, \ldots, \lambda_{N_0}) \in \mathbb{K}^{N_0 \times N_0}$$
$$B = (b_{n,k})_{1 \leq n \leq N_0, 1 \leq k \leq m} \in \mathbb{K}^{N_0 \times m}$$

with $b_{n,k} = \langle (\mathcal{A} - \lambda_n I_{\mathcal{H}}) L e_k, \psi_n \rangle_{\mathcal{H}}$ and $(e_1, e_2, \ldots, e_m)$ the canonical basis of $\mathbb{K}^m$,

$$Y(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_{N_0}(t) \end{bmatrix} = \begin{bmatrix} \langle X(t), \psi_1 \rangle_{\mathcal{H}} \\ \vdots \\ \langle X(t), \psi_{N_0} \rangle_{\mathcal{H}} \end{bmatrix}, \quad P(t) = \begin{bmatrix} \langle p(t), \psi_1 \rangle_{\mathcal{H}} \\ \vdots \\ \langle p(t), \psi_{N_0} \rangle_{\mathcal{H}} \end{bmatrix}$$

A3) We assume that (A, B) is stabilizable.

Closed-loop dynamics and stability result

Closed-loop system dynamics with predictor feedback synthesized based on the truncated model:

$$\begin{aligned}\frac{\mathrm{d}X}{\mathrm{d}t}(t) &= \mathcal{A}X(t) + p(t),\\ \mathcal{B}X(t) &= KY(t),\\ X(0) &= X_0\end{aligned}$$

with gain $K \in \mathbb{K}^{m \times N_0}$ such that $A_{\mathrm{cl}} \triangleq A + BK$ is Hurwitz.

Stability result

There exist constants $\kappa, C_1, C_2 > 0$ such that

$$\|X(t)\|_{\mathcal{H}} + \|u(t)\| \leq C_1 e^{-\kappa t} \|X_0\|_{\mathcal{H}} + C_2 \sup_{\tau \in [0,t]} \|p(\tau)\|_{\mathcal{H}}$$

Sharp introduction to the concept of predictor feedback

Objective: stabilization of LTI plants in the presence of an input delay $D > 0$:

$$\dot{x}(t) = Ax(t) + Bu(t - D), \quad t \geq 0,$$

for a stabilizable pair (A, B).

Idea: setting $u(t - D) = Kx(t)$ we have:

$$\dot{x}(t) = A_{\mathrm{cl}}x(t)$$

where K is selected such that $A_{\mathrm{cl}} = A + BK$ is Hurwitz.

Predictor component: the control input at time t takes the form of $u(t) = Kx(t + D)$; we need to predict $x(t + D)$ from $x(t)$:

$$x(t + D) = e^{DA} \left\{ x(t) + \int_{t-D}^{t} e^{(t-D-s)A} Bu(s) \, \mathrm{d}s \right\}.$$

Reference: seminal work [Artstein, 1982].

Extension to diagonal infinite-dimensional systems?

Positive answer for the reaction-diffusion system:

$$\begin{aligned} y_t &= y_{xx} + c(x)y \\ y(t,0) &= 0, \quad y(t,L) = u(t-D) \\ y(0,x) &= y_0(x) \end{aligned}$$

reported in [Prieur and Trélat, 2018] for a constant and known input delay $D > 0$.

Possible extension to:

- General Sturm-Liouville operator?
- Dirichlet/Neumann/Robin boundary condition and boundary control?
- Robustness issues:
  - Uncertain and time-varying input delay $D(t)$?
  - Boundary and distributed perturbations?
- Extension to diagonal infinite-dimensional systems?

2 Stabilization with delayed boundary control

- Case of a constant and known input delay
- Case of an uncertain and time-varying input delay
- Extensions

Problem setting

$\mathcal{H}$ is a separable Hilbert space on $\mathbb{K}$, which is either $\mathbb{R}$ or $\mathbb{C}$.

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t}(t) &= \mathcal{A}X(t) + p(t), && t \geq 0 \\ \mathcal{B}X(t) &= u(t - D), && t \geq 0 \\ X(0) &= X_0 \end{aligned}$$

Assumptions:

- $(\mathcal{A}, \mathcal{B})$ is a boundary control system.
- Assumption A1 holds: the disturbance free operator $\mathcal{A}_0$ is diagonal in a Riesz basis.
- Assumption A2 holds: $\mathcal{A}_0$ admits a finite number of unstable modes while the real part of the stable ones do not accumulate at 0.
- The control input $u(t) \in \mathbb{K}^m$ is subject to a constant and known delay $D > 0$.

Finite dimensional truncated model

$$\dot{Y}(t) = AY(t) + Bu(t - D) + P(t),$$

where

$$A = \operatorname{diag}(\lambda_1, \ldots, \lambda_{N_0}) \in \mathbb{K}^{N_0 \times N_0}$$
$$B = (b_{n,k})_{1 \leq n \leq N_0, 1 \leq k \leq m} \in \mathbb{K}^{N_0 \times m}$$

with $b_{n,k} = \langle (\mathcal{A} - \lambda_n I_{\mathcal{H}}) L e_k, \psi_n \rangle_{\mathcal{H}}$ and $(e_1, e_2, \ldots, e_m)$ the canonical basis of $\mathbb{K}^m$,

$$Y(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_{N_0}(t) \end{bmatrix} = \begin{bmatrix} \langle X(t), \psi_1 \rangle_{\mathcal{H}} \\ \vdots \\ \langle X(t), \psi_{N_0} \rangle_{\mathcal{H}} \end{bmatrix}, \quad P(t) = \begin{bmatrix} \langle p(t), \psi_1 \rangle_{\mathcal{H}} \\ \vdots \\ \langle p(t), \psi_{N_0} \rangle_{\mathcal{H}} \end{bmatrix}$$

A3) We assume that (A, B) is stabilizable.

Closed-loop dynamics and main result

Closed-loop system dynamics with predictor feedback synthesized based on the truncated model:

$$\frac{\mathrm{d}X}{\mathrm{d}t}(t) = \mathcal{A}X(t) + p(t),$$
$$\mathcal{B}X(t) = u(t - D),$$
$$u(t) = \varphi(t) K \left\{ Y(t) + \int_{\max(t-D,0)}^{t} e^{(t-s-D)A} B u(s) \,\mathrm{d}s \right\},$$
$$X(0) = X_0$$

with gain $K \in \mathbb{K}^{m \times N_0}$ such that $A_{\mathrm{cl}} \triangleq A + e^{-DA}BK$ is Hurwitz.

Stability result [H. Lhachemi and Prieur, 2021]

There exist constants $\kappa, C_1, C_2 > 0$ such that

$$\|X(t)\|_{\mathcal{H}} + \|u(t)\| \leq C_1 e^{-\kappa t} \|X_0\|_{\mathcal{H}} + C_2 \sup_{\tau \in [0,t]} \|p(\tau)\|_{\mathcal{H}}$$

Sketch of proof

Proof based on the Lyapunov functional:

$$\begin{aligned} V(t) = \gamma_1 \left\{ Z(t)^* P Z(t) + \int_{t-D}^{t} \varphi(s) Z(s)^* P Z(s) \, \mathrm{d}s \right\} \\ + \gamma_2 \varphi(t-D) Z(t-D)^* P Z(t-D) \\ + \frac{1}{2} \sum_{k \geq N_0+1} \left| \langle X(t) - Bu(t-D), \psi_k \rangle_{\mathcal{H}} \right|^2 , \end{aligned}$$

where (Artstein transformation [Artstein, 1982])

$$Z(t) \triangleq Y(t) + \int_{t-D}^{t} e^{(t-s-D)A} Bu(s) \, \mathrm{d}s$$

with $P \succ 0$ such that $A_{\mathrm{cl}}^* P + P A_{\mathrm{cl}} = -I_{N_0}$ and $\gamma_1, \gamma_2 > 0$ are sufficiently large constants.

2 Stabilization with delayed boundary control

- Case of a constant and known input delay
- Case of an uncertain and time-varying input delay
- Extensions

Problem setting

$\mathcal{H}$ is a separable Hilbert space on $\mathbb{K}$, which is either $\mathbb{R}$ or $\mathbb{C}$.

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t}(t) &= \mathcal{A}X(t), & t \geq 0 \\ \mathcal{B}X(t) &= u(t - D(t)), & t \geq 0 \\ X(0) &= X_0 \end{aligned}$$

Assumptions:

- $(\mathcal{A}, \mathcal{B})$ is a boundary control system.
- Assumption A1 holds: the disturbance free operator $\mathcal{A}_0$ is diagonal in a Riesz basis.
- Assumption A2 holds: $\mathcal{A}_0$ admits a finite number of unstable modes while the real part of the stable ones do not accumulate at 0.
- The control input $u(t) \in \mathbb{K}^m$ is subject to an uncertain and time-varying delay $D(t) > 0$.

Finite dimensional truncated model

$$\dot{Y}(t) = AY(t) + Bu(t - D(t)),$$

where

$$A = \mathrm{diag}(\lambda_1, \dots, \lambda_{N_0}) \in \mathbb{K}^{N_0 \times N_0}$$
$$B = (b_{n,k})_{1 \leq n \leq N_0, 1 \leq k \leq m} \in \mathbb{K}^{N_0 \times m}$$

with $b_{n,k} = \langle (\mathcal{A} - \lambda_n I_{\mathcal{H}}) L e_k, \psi_n \rangle_{\mathcal{H}}$ and $(e_1, e_2, \dots, e_m)$ the canonical basis of $\mathbb{K}^m$,

$$Y(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_{N_0}(t) \end{bmatrix} = \begin{bmatrix} \langle X(t), \psi_1 \rangle_{\mathcal{H}} \\ \vdots \\ \langle X(t), \psi_{N_0} \rangle_{\mathcal{H}} \end{bmatrix}$$

A3) We assume that (A, B) is stabilizable.

Robustness of constant delay predictor feedback

$$\dot{x}(t) = Ax(t) + Bu(t - D(t)), \quad t \geq 0,$$

with $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ such that (A, B) is stabilizable.

Uncertain and time-varying input delay $D \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{R}_+)$.

We assume that there exist known constants $D_0 > 0$ and $0 < \delta < D_0$ such that $|D(t) - D_0| \leq \delta$.

Constant-delay linear predictor feedback:

$$u(t) = K \left\{ x(t) + \int_{t-D_0}^{t} e^{(t-D_0-s)A} Bu(s) \, \mathrm{d}s \right\}$$

where $K \in \mathbb{R}^{m \times n}$ is such that $A_{\mathrm{cl}} = A + e^{-D_0 A} BK$ is Hurwitz.

Sufficient condition on $\delta > 0$ such that the closed-loop system is stable?

Preliminary Lemma

The following preliminary Lemma is a variation of [Fridman, 2006].

Lemma

Let $M, N \in \mathbb{R}^{n\times n}$, $D_0 > 0$, and $\delta \in (0, D_0)$ be given. Assume that there exist $\kappa > 0$, $P_1, Q \in \mathbb{S}_n^{+*}$, and $P_2, P_3 \in \mathbb{R}^{n\times n}$ such that $\Theta(\delta,\kappa) \preceq 0$ with

$$\Theta(\delta,\kappa) = \begin{bmatrix} 2\kappa P_1 + M^\top P_2 + P_2^\top M & P_1 - P_2^\top + M^\top P_3 & \delta P_2^\top N \\ P_1 - P_2 + P_3^\top M & -P_3 - P_3^\top + 2\delta Q & \delta P_3^\top N \\ \delta N^\top P_2 & \delta N^\top P_3 & -\delta e^{-2\kappa D_0} Q \end{bmatrix}.$$

Then, there exists $C_0 > 0$ such that, for any $D \in \mathcal{C}^0(\mathbb{R}_+;\mathbb{R}_+)$ with $|D - D_0| \leq \delta$, the trajectory x of:

$$\dot{x}(t) = Mx(t) + N\left\{x(t - D(t)) - x(t - D_0)\right\};$$
$$x(\tau) = x_0(\tau),\ \tau \in [-D_0 - \delta, 0]$$

with initial condition $x_0 \in W$ satisfies $\|x(t)\| \leq C_0 e^{-\kappa t}\|x_0\|_W$ for all $t \geq 0$.

Sketch of proof

We define $V(t) = V_1(t) + V_2(t)$ with $V_1(t) = x(t)^\top P_1 x(t)$ and

$$V_2(t) = \int_{-D_0-\delta}^{-D_0+\delta} \int_{t+\theta}^{t} e^{2\kappa(s-t)} \dot{x}(s)^\top Q \dot{x}(s) \,\mathrm{d}s \,\mathrm{d}\theta$$

where $P_1, Q \in \mathbb{S}_n^{+*}$.

We have the inequalities:

$$\lambda_{\mathrm{m}}(P_1)\|x(t)\|^2 \leq V(t) \leq \max\left(\lambda_{\mathrm{M}}(P_1), 2\delta\lambda_{\mathrm{M}}(Q)\right) \|x(t+\cdot)\|_W^2$$

The computation of the time derivative of V yields

$$\begin{aligned}\dot{V}(t) &= 2x(t)^\top P_1 \dot{x}(t) + 2\delta \dot{x}(t)^\top Q \dot{x}(t) - 2\kappa V_2(t) \\ &- \int_{-D_0-\delta}^{-D_0+\delta} e^{2\kappa\theta} \dot{x}(t+\theta)^\top Q \dot{x}(t+\theta) \,\mathrm{d}\theta\end{aligned}$$

Sketch of proof

Introducing $P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}$ with the slack variables $P_2, P_3 \in \mathbb{R}^{n\times n}$:

$$\dot{V}(t) + 2\kappa V(t) \leq \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}^\top \Psi \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix},$$

where

$$\Psi \triangleq P^\top \begin{bmatrix} 0 & I \\ M & -I \end{bmatrix} + \begin{bmatrix} 0 & I \\ M & -I \end{bmatrix}^\top P + 2 \begin{bmatrix} \kappa P_1 & 0 \\ 0 & \delta Q \end{bmatrix}$$
$$+ \delta e^{2\kappa D_0} P^\top \begin{bmatrix} 0 \\ N \end{bmatrix} Q^{-1} \begin{bmatrix} 0 \\ N \end{bmatrix}^\top P.$$

From $\Theta(\delta, \kappa) \preceq 0$, the use of the Schur complement yields $\dot{V}(t) + 2\kappa V(t) \leq 0$.

Useful "converse" result

The conclusions of the previous Lemma imply that the matrix M is Hurwitz. A form of "converse" result is provided below.

Lemma

Let $M, N \in \mathbb{R}^{n \times n}$ with M Hurwitz and $D_0 > 0$ be given. Then there exist $\delta \in (0, D_0)$ and $\kappa > 0$ such that the LMI $\Theta(\delta, \kappa) \prec 0$ is feasible.

Hence M Hurwitz implies the existence of small enough deviations of the delay around its nominal value such that the system is exponentially stable.

Robustness of predictor feedback

Theorem [Lhachemi, Prieur, and Shorten, 2019]

Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ with (A, B) stabilizable. Let $D_0 > 0$ and let φ be a transition signal over $[0, t_0]$ with $t_0 > 0$. Let $K \in \mathbb{R}^{m \times n}$ be such that $A_{\mathrm{cl}} \triangleq A + e^{-D_0 A} BK$ is Hurwitz. Then, there exist $\delta \in (0, D_0)$ such that for any $D \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{R}_+)$ with $|D - D_0| \leq \delta$,

$$\dot{x}(t) = Ax(t) + Bu(t - D(t)),$$
$$u(t) = \varphi(t) K \left\{ x(t) + \int_{t-D_0}^{t} e^{(t-D_0-s)A} Bu(s) \, \mathrm{d}s \right\},$$

with initial condition $x(0) = x_0 \in \mathbb{R}^n$ is exponentially stable:

$$\|x(t)\| + \|u(t)\| \leq Ce^{-\kappa t} \|x_0\|, \quad \forall t \geq 0.$$

The above conclusion holds true for any $\delta \in (0, D_0)$ and any $\kappa > 0$ such that the LMI $\Theta(\delta, \kappa) \preceq 0$ is feasible with $M = A_{\mathrm{cl}}$ and $N = BK$.

Sketch of proof

The introduction of the Artstein transformation

$$z(t) = x(t) + \int_{t-D_0}^{t} e^{(t-D_0-s)A} B u(s) \,\mathrm{d}s$$

yields, for times $t \geq t_0 + D_0 + \delta$,

$$\dot{z}(t) = A_{\mathrm{cl}} z(t) + BK\{z(t - D(t)) - z(t - D_0)\}$$

with $A_{\mathrm{cl}} = A + e^{-D_0 A} BK$ Hurwitz.

The claimed conclusion easily follows from the preliminary lemma.

Application to diagonal infinite-dimensional systems

Predictor feedback synthesized based on the truncated model:

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t}(t) &= \mathcal{A}X(t), \\ \mathcal{B}X(t) &= u(t - D(t)), \\ u(t) &= \varphi(t)K\left\{ Y(t) + \int_{\max(t-D_0,0)}^{t} e^{(t-s-D_0)A} Bu(s)\,\mathrm{d}s \right\}, \\ X(0) &= X_0 \end{aligned}$$

with gain $K \in \mathbb{K}^{m\times N_0}$ such that $A_{\mathrm{cl}} \triangleq A + e^{-D_0 A}BK$ is Hurwitz.

Stability result [Lhachemi, Prieur, and Shorten, 2019]

There exist $\delta, \eta > 0$ such that, for any $\delta_r > 0$, there exists $C > 0$ such that for any $X_0 \in D(\mathcal{A}_0)$ and $D \in \mathcal{C}^2(\mathbb{R}_+;\mathbb{R}_+)$ with $|D - D_0| \leq \delta$ and $|\dot{D}| \leq \delta_r$,

$$\|X(t)\|_{\mathcal{H}} + \|u(t)\| \leq Ce^{-\eta t}\|X_0\|_{\mathcal{H}}$$

Numerical example

Consider the following reaction-diffusion equation:

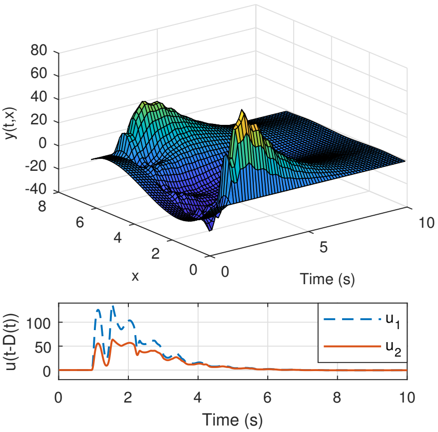
$$\begin{cases} y_t(t,x) = a y_{xx}(t,x) + c y(t,x), & (t,x) \in \mathbb{R}_+ \times (0,L) \\ \begin{bmatrix} y(t,0) \\ y(t,L) \end{bmatrix} = u(t - D(t)), & t > 0 \end{cases}$$

Numerical setting:

- system parameters: $a = c = 0.5$, $L = 2\pi$, $D_0 = 1\,\mathrm{s}$;
- first eigenvalues: $\lambda_1 = 0.375$, $\lambda_2 = 0$, $\lambda_3 = -0.625$, $\lambda_4 = -1.5$;
- control design: $N_0 = 3$, gain $K \in \mathbb{R}^{2\times 3}$ is computed to place the poles of the closed-loop truncated model at -0.75, -1, and -1.25.

Application of the main theorem: exponential stability of the closed-loop system with decay rate $\kappa = 0.2$ for $\delta = 0.260$.

Numerical example



Delay: $D(t) = 1 + 0.25\sin(3\pi t + \pi/4)$

2 Stabilization with delayed boundary control

- Case of a constant and known input delay
- Case of an uncertain and time-varying input delay
- Extensions

Extension 1: ISS w.r.t. boundary disturbances

Closed-loop system dynamics with boundary disturbances d_1, d_2:

$$\frac{\mathrm{d}X}{\mathrm{d}t}(t) = \mathcal{A}X(t),$$
$$\mathcal{B}X(t) = u(t - D(t)) + d_1(t),$$
$$u(t) = \varphi(t) \left\{ KY(t) + K \int_{\max(t-D_0,0)}^{t} e^{(t-s-D_0)A} Bu(s) \,\mathrm{d}s + d_2(t) \right\},$$
$$X(0) = X_0$$

with gain $K \in \mathbb{K}^{m \times N_0}$ such that $A_{\mathrm{cl}} \triangleq A + e^{-D_0 A} BK$ is Hurwitz.

Stability result [Lhachemi, Shorten, and Prieur, 2020]

Assume in addition that $\sup_{n \geq N_0+1} |\lambda_n / \operatorname{Re} \lambda_n| < +\infty$. Then there exist constants $\delta, \kappa, C_i > 0$ such that, for any $X_0 \in \mathcal{H}$, $D \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{R}_+)$ with $|D - D_0| \leq \delta$, and $d_i \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{K}^m)$,

$$\|X(t)\|_{\mathcal{H}} + \|u(t)\| \leq C_1 e^{-\kappa t} \|X_0\|_{\mathcal{H}} + C_2 \sup_{\tau \in [0,t]} \|(d_1(\tau), d_2(\tau))\|$$

Extension 2: distinct input delays

Case of distinct uncertain and time-varying input delays $D_k(t)$:

$$\frac{\mathrm{d}X}{\mathrm{d}t}(t) = \mathcal{A}X(t),$$

$$\mathcal{B}X(t) = \tilde{u}(t) = (u_1(t - D_1(t)), \dots, u_m(t - D_m(t))),$$

$$u(t) = \varphi(t)K\left\{Y(t) + \sum_{i=1}^{m} \int_{t-D_{0,i}}^{t} e^{(t-D_{0,i}-s)A_{N_0}} B_{N_0,i} u_i(s)\,\mathrm{d}s\right\},$$

$$X(0) = X_0,$$

with $K_k \in \mathbb{K}^{1\times N_0}$ such that $A_{\mathrm{cl}} = A_{N_0} + \sum_{k=1}^{m} e^{-D_{0,k}A_{N_0}} B_{N_0,k} K_k$ is Hurwitz.

Stability result [Lhachemi, Prieur, and Shorten, 2020]

There exist $\delta_k, \eta > 0$ such that, for any $\delta_r > 0$, there exists $C > 0$ such that for any $X_0 \in D(\mathcal{A}_0)$ and $D_k \in \mathcal{C}^2(\mathbb{R}_+;\mathbb{R}_+)$ with $|D_k - D_{0,k}| \leq \delta_k$ and $|\dot{D}_k| \leq \delta_r$,

$$\|X(t)\|_{\mathcal{H}} + \|u(t)\| \leq Ce^{-\eta t}\|X_0\|_{\mathcal{H}}$$

Problem setting

Let $a > 0$, let $b, c \in \mathbb{R}$, and let $\theta_1, \theta_2 \in [0, 2\pi)$ be arbitrary.

$$\begin{aligned} & y_t(t,x) = a y_{xx}(t,x) + b y(t,x) + c y(t - h(t), x) + p(t,x) \\ & \cos(\theta_1) y(t,0) - \sin(\theta_1) y_x(t,0) = u_1(t) \\ & \cos(\theta_2) y(t,1) + \sin(\theta_2) y_x(t,1) = u_2(t) \\ & y(\tau, x) = \phi(\tau, x), \quad \tau \in [-h_M, 0] \end{aligned}$$

$t \geq 0$, $x \in (0,1)$.

- $y(t,\cdot) \in L^2(0,1)$ is the state at time t;
- $u_1(t), u_2(t) \in \mathbb{R}$ are the control inputs
  - $\Rightarrow$ with possibly one single control input (i.e., either $u_1 = 0$ or $u_2 = 0$);
- $p \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(0,1))$ is a distributed disturbance;
- $h \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{R}_+)$ with $0 < h_m \leq h(t) \leq h_M$ is a time-varying delay;
- $\phi \in \mathcal{C}^0([-h_M, 0]; L^2(0,1))$ is the initial condition.

3 Boundary stabilization in the presence of a state-delay

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Equivalent representation

We rewrite the reaction-diffusion system under the form:

$$\begin{aligned} y_t(t,x) &= a y_{xx}(t,x) + (b+c)y(t,x) \\ &\quad + c\left\{y(t-h(t),x) - y(t,x)\right\} + p(t,x) \\ \cos(\theta_1)y(t,0) &- \sin(\theta_1)y_x(t,0) = u_1(t) \\ \cos(\theta_2)y(t,1) &+ \sin(\theta_2)y_x(t,1) = u_2(t) \\ y(\tau,x) &= \phi(\tau,x), \quad \tau \in [-h_M, 0] \end{aligned}$$

Interpretation:

- $cy(t,x)$ is viewed as the "nominal contribution" of the term $cy(t-h(t),x)$;
- $c\left\{y(t-h(t),x) - y(t,x)\right\}$ is viewed as a "disturbance term" introduced by the occurrence of the delay $h(t)$.

Abstract formulation of the problem

We define $X(t) = y(t, \cdot)$, $\mathcal{H} = L^2(0,1)$, and $\mathcal{A}f = af'' + (b+c)f$ and $\mathcal{B}f = (\cos(\theta_1)f(0) - \sin(\theta_1)f'(0), \cos(\theta_2)f(1) + \sin(\theta_2)f'(1)) \in \mathbb{R}^2$ defined on $D(\mathcal{A}) = D(\mathcal{B}) = H^2(0,1)$.

$$\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}t}(t) &= \mathcal{A}X(t) + c\{X(t-h(t)) - X(t)\} + p(t), && t \geq 0 \\ \mathcal{B}X(t) &= u(t) = (u_1(t), u_2(t)), && t \geq 0 \\ X(\tau) &= \Phi(\tau), && \tau \in [-h_M, 0] \end{aligned}$$

Key properties: $\mathcal{A}_0$ is self-adjoint, has compact resolvent, and has simple eigenvalues. Hence we have a Hilbert basis $(e_n)_{n\geq 1}$ of $L^2(0,L)$ consisting of eigenfunctions of $\mathcal{A}_0$ associated with the sequence of simple real eigenvalues

$$-\infty < \cdots < \lambda_n < \cdots < \lambda_1$$

Spectral reduction of the problem

Introducing the coefficients of projection $x_n(t) = \langle X(t), e_n \rangle$, the system trajectory can be expanded as a series in the eigenfunctions e_n, convergent in $L^2(0,1)$,

$$X(t) = \sum_{n \geq 1} x_n(t) e_n.$$

Equivalent infinite-dimensional control system:

$$\begin{aligned}\dot{x}_n(t) = \lambda_n x_n(t) &+ c \left\{ x_n(t-h(t)) - x_n(t) \right\} \\ &+ \langle (\mathcal{A} - \lambda_n) L u(t), e_n \rangle + \langle p(t), e_n \rangle\end{aligned}$$

$n \geq 1$, with

$$\|X(t)\|^2 = \sum_{n \geq 1} |x_n(t)|^2.$$

Finite dimensional truncated model

For a number of modes $N_0 \geq 0$ to be determined latter:

$$\dot{Y}(t) = AY(t) + c\{Y(t - h(t)) - Y(t)\} + Bu(t) + P(t),$$

where

$$A = \operatorname{diag}(\lambda_1, \ldots, \lambda_{N_0}) \in \mathbb{R}^{N_0 \times N_0}$$
$$B = (b_{n,k})_{1 \leq n \leq N_0, 1 \leq k \leq 2} \in \mathbb{R}^{N_0 \times 2}$$

with $b_{n,k} = \langle (\mathcal{A} - \lambda_n) L f_k, e_n \rangle_{\mathcal{H}}$ and (f_1, f_2) the canonical basis of $\mathbb{R}^2$,

$$Y(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_{N_0}(t) \end{bmatrix} = \begin{bmatrix} \langle X(t), e_1 \rangle_{\mathcal{H}} \\ \vdots \\ \langle X(t), e_{N_0} \rangle_{\mathcal{H}} \end{bmatrix}, \quad P(t) = \begin{bmatrix} \langle p(t), e_1 \rangle_{\mathcal{H}} \\ \vdots \\ \langle p(t), e_{N_0} \rangle_{\mathcal{H}} \end{bmatrix}$$

Representation for control design and stability analysis

Final representation of the reaction-diffusion equation for control design and stability analysis:

$$\begin{aligned} \dot{Y}(t) &= AY(t) + c\{Y(t-h(t)) - Y(t)\} + Bu(t) + P(t) \\ \dot{x}_n(t) &= \lambda_n x_n(t) + c\,\{x_n(t-h(t)) - x_n(t)\} \\ &\quad + \langle (\mathcal{A} - \lambda_n) Lu(t), e_n \rangle + \langle p(t), e_n \rangle \end{aligned}$$

with $n \geq N_0 + 1$.

Two-step control design strategy:

1. Select the number N_0 of modes captured by the truncated model to ensure the exponential stability of the residual dynamics.
2. For an arbitrarily given number of modes N_0, design a feedback law ensuring the exponential stability of the truncated model.

3 Boundary stabilization in the presence of a state-delay

- Spectral reduction
- **Control design on the truncated model**
- Stability assessment of the infinite-dimensional system
- Numerical application

Control strategy for the finite-dimensional truncated model

Truncated model for an arbitrarily given number of modes N_0:

$$\dot{Y}(t) = AY(t) + c\{Y(t - h(t)) - Y(t)\} + Bu(t) + P(t)$$

Lemma

The pair (A, B) satisfies the Kalman condition.

($\Rightarrow$ also holds in the case of one single boundary control input)

Setting

$$u(t) = KY(t)$$

we have

$$\dot{Y}(t) = A_{\mathrm{cl}} Y(t) + c\{Y(t - h(t)) - Y(t)\} + P(t)$$

with $A_{\mathrm{cl}} = A + BK$ Hurwitz.

Stability of the closed-loop truncated model

Lemma (truncated model)

Let $N_0 \geq 1$ and $0 < h_m < h_M$ be arbitrarily given. Let $K \in \mathbb{R}^{2 \times N_0}$ be such that $A_{\mathrm{cl}} = A + BK$ is Hurwitz with simple eigenvalues $\mu_1, \ldots, \mu_{N_0} \in \mathbb{C}$ and $\operatorname{Re} \mu_n < -3|c|$ for all $1 \leq n \leq N_0$. Then, there exist constants $\sigma, C_2, C_3 > 0$ such that, for all $Y_\Phi \in \mathcal{C}^0([-h_M, 0]; \mathbb{R}^{N_0})$, $h \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{R})$ with $h_m \leq h \leq h_M$, and $P \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^{N_0})$, the trajectory $Y(t)$ of the truncated model with command input $u(t) = KY(t)$ satisfies

$$\|Y(t)\| \leq C_2 e^{-\sigma t} \sup_{\tau \in [-h_M, 0]} \|Y_\Phi(\tau)\| + C_3 \operatorname*{ess\,sup}_{\tau \in [0,t]} e^{-\sigma(t-\tau)} \|P(\tau)\|.$$

Sketch of proof

As the eigenvalues of A_{cl} are simple, there exists $Q \in \mathbb{C}^{N_0 \times N_0}$ such that $Q A_{\mathrm{cl}} Q^{-1} = \Lambda \triangleq \mathrm{diag}(\mu_1, \ldots, \mu_{N_0})$.

With $Z(t) = QY(t)$ and $\hat{P}(t) = QP(t)$, we obtain:

$$\dot{Z}(t) = \Lambda Z(t) + c\left\{Z(t-h(t)) - Z(t)\right\} + \hat{P}(t).$$

Introducing $v(t) = Z(t) - Z(t-h(t))$, successive estimates yield

$$\begin{aligned} \sup_{\tau\in[h_M,t]} e^{\sigma\tau}\|v(\tau)\| \leq\ & 2e^{\sigma h_M}\|Z_\Phi(0)\| + \delta \sup_{\tau\in[0,h_M]} e^{\sigma\tau}\|v(\tau)\| \\ & + \delta \sup_{\tau\in[h_M,t]} e^{\sigma\tau}\|v(\tau)\| + \frac{\delta}{|c|} \operatorname*{ess\,sup}_{\tau\in[0,t]} e^{\sigma\tau}\|\hat{P}(\tau)\| \end{aligned}$$

for all $t \geq h_M$ with $\alpha = -\max_{1\leq n\leq N_0} \operatorname{Re}\mu_n > 3|c|$, $\sigma \in (0,\alpha)$ arbitrary, and

$$\delta = \frac{|c|}{\alpha-\sigma}\left\{1 + 2e^{\sigma h_M}\right\} \underset{\sigma\to 0^+}{\sim} \frac{3|c|}{\alpha} < 1.$$

Sketch of proof

Selecting $\sigma \in (0, \alpha)$ small enough such that $\delta < 1$, we infer

$$\sup_{\tau \in [h_M, t]} e^{\sigma\tau} \| v(\tau) \| \leq \frac{2e^{\sigma h_M}}{1-\delta} \| Z_\Phi(0) \| + \frac{\delta}{1-\delta} \sup_{\tau \in [0, h_M]} e^{\sigma\tau} \| v(\tau) \| + \frac{\delta}{|c|(1-\delta)} \operatorname*{ess\,sup}_{\tau \in [0,t]} e^{\sigma\tau} \| \hat{P}(\tau) \|$$

for all $t \geq h_M$.

The conclusion follows by 1) estimating $\sup_{\tau \in [0, h_M]} e^{\sigma\tau} \| v(\tau) \|$; 2) using the estimate:

$$\sup_{\tau \in [0,t]} e^{\sigma\tau} \| Z(\tau) \| \leq \| Z_\Phi(0) \| + \frac{|c|}{\alpha - \sigma} \sup_{\tau \in [0,t]} e^{\sigma\tau} \| v(\tau) \| + \frac{1}{\alpha - \sigma} \operatorname*{ess\,sup}_{\tau \in [0,t]} e^{\sigma\tau} \| \hat{P}(\tau) \|$$

for all $t \geq 0$; and 3) $Y(t) = Q^{-1} Z(t)$.

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Stability of the infinite-dimensional residual dynamics

Lemma (residual infinite-dimensional dynamics)

Let $0 < h_m < h_M$ and $\sigma, C_4, C_5 > 0$ be arbitrarily given. Let $N_0 \geq 1$ be such that $\lambda_{N_0+1} < -2\sqrt{5}|c|$. Then, there exist constants $\kappa \in (0, \sigma)$ and $C_6, C_7 > 0$ such that, for all $\Phi \in \mathcal{C}^0([-h_M, 0]; \mathcal{H})$, $p \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathcal{H})$, $h \in \mathcal{C}^0(\mathbb{R}_+; \mathbb{R})$ with $h_m \leq h \leq h_M$, and $u \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+; \mathbb{R}^2)$ with

$$\|u(t)\| + \|\dot{u}(t)\| \leq C_4 e^{-\sigma t} \sup_{\tau \in [-h_M, 0]} \|\Phi(\tau)\| + C_5 \operatorname*{ess\,sup}_{\tau \in [0,t]} e^{-\sigma(t-\tau)} \|p(\tau)\|,$$

we have

$$\sum_{n \geq N_0+1} |x_n(t)|^2 \leq C_6 e^{-2\kappa t} \sup_{\tau \in [-h_M, 0]} \|\Phi(\tau)\|^2 + C_7 \operatorname*{ess\,sup}_{\tau \in [0,t]} e^{-2\kappa(t-\tau)} \|p(\tau)\|^2.$$

Sketch of proof

Introducing $z_n(t) = \langle X(t) - Lu(t), e_n \rangle = x_n(t) - \langle Lu(t), e_n \rangle$ and

$$V(t) = \sum_{n \geq N_0+1} |z_n(t) - z_n(t-h(t))|^2,$$

successive estimates yield, for $t \geq 2h_M$,

$$\begin{aligned} \sup_{\tau \in [2h_M, t]} e^{2\kappa\tau} V(\tau) \leq\ & 16 e^{4\kappa h_M} Z(h_M) + \eta \sup_{\tau \in [h_M, 2h_M]} e^{2\kappa\tau} V(\tau) \\ & + \eta \sup_{\tau \in [2h_M, t]} e^{2\kappa\tau} V(\tau) + \frac{\gamma_1 \eta}{|c|^2} \sup_{\tau \in [-h_M, 0]} \|\Phi(\tau)\|^2 \\ & + \frac{(1+\gamma_2)\eta}{|c|^2} \operatorname*{ess\,sup}_{\tau \in [0,t]} e^{2\kappa\tau} \|p(\tau)\|^2. \end{aligned}$$

with $\beta = -\lambda_{N_0+1}/2 > \sqrt{5}|c|$ and

$$\eta = \frac{|c|^2}{\beta(\beta-\kappa)} \left\{ 1 + 4e^{2\kappa h_M} \right\} \underset{\kappa \to 0^+}{\sim} \frac{5|c|^2}{\beta^2} < 1.$$

Stability of the closed-loop reaction-diffusion equation

Theorem [Lhachemi and Shorten, 2020]

Let $0 < h_m < h_M$ be arbitrarily given. Let $N_0 \geq 1$ be such that $\lambda_{N_0+1} < -2\sqrt{5}|c|$. Let $K \in \mathbb{R}^{2\times N_0}$ be such that $A_{\mathrm{cl}} = A + BK$ is Hurwitz with simple eigenvalues $\mu_1, \dots, \mu_{N_0} \in \mathbb{C}$ satisfying $\operatorname{Re}\mu_n < -3|c|$ for all $1 \leq n \leq N_0$. Then, there exist constants $\kappa, C_0, C_1 > 0$ such that, for any initial condition $\phi \in \mathcal{C}^0([-h_M, 0]; L^2(0,1))$, any distributed perturbation $p \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; L^2(0,1))$, and any delay $h \in \mathcal{C}^0(\mathbb{R}_+;\mathbb{R})$ with $h_m \leq h \leq h_M$, the state-delayed reaction diffusion equation with $u = KY$ satisfies

$$\|y(t,\cdot)\| \leq C_0 e^{-\kappa t} \sup_{\tau\in[-h_M,0]} \|\phi(\tau,\cdot)\| + C_1 \operatorname*{ess\,sup}_{\tau\in[0,t]} e^{-\kappa(t-\tau)}\|p(\tau,\cdot)\|$$

for all $t \geq 0$.

3 Boundary stabilization in the presence of a state-delay

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Numerical application

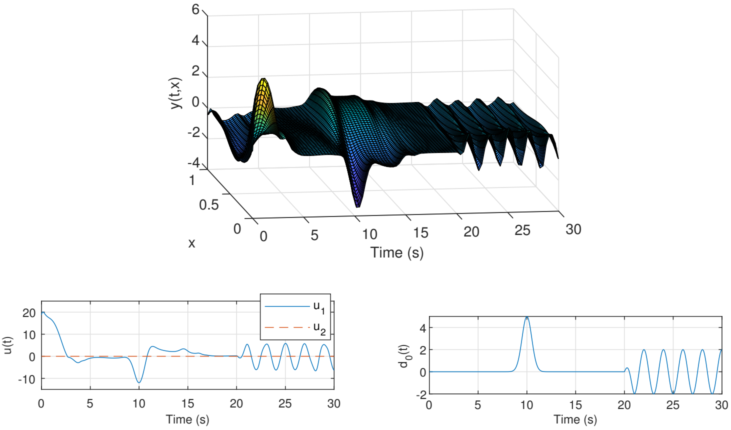
$$
\begin{aligned}
& y_t(t,x) = a y_{xx}(t,x) + b y(t,x) + c y(t-h(t),x) + p(t,x) \\
& \cos(\theta_1) y(t,0) - \sin(\theta_1) y_x(t,0) = u_1(t) \\
& \cos(\theta_2) y(t,1) + \sin(\theta_2) y_x(t,1) = u_2(t) \\
& y(\tau,x) = \phi(\tau,x), \quad \tau \in [-h_M, 0]
\end{aligned}
$$

$t \geq 0$, $x \in (0,1)$.

Numerical setting:

- system parameters: $a = 0.2$, $b = 2$, $c = 1$, $\theta_1 = \pi/3$, and $\theta_2 = \pi/10$;
- first eigenvalues: $\lambda_1 \approx 2.5561$, $\lambda_2 \approx -0.1186 > -2\sqrt{5}|c|$, and $\lambda_3 \approx -6.2299 < -2\sqrt{5}|c|$;
- control design: $N_0 = 2$, gain $K \in \mathbb{R}^{2\times 2}$ is computed to place the poles of the closed-loop truncated model at $\mu_1 = -3.5$ and $\mu_2 = -4$ with in particular $\mu_2 < \mu_1 < -3|c|$;

Numerical example



- Distributed disturbance: $p(t,x) = d_0(t)(1-x)$.
- Initial condition:
  $\Phi(t,x) = (1-t)^2 \left\{(1-2x)/2 + 20x(1-x)(x-3/5)\right\}$.
- Delay: $h(t) = 2 + 1.5\sin(t)$.

PI regulation of infinite-dimensional systems

PI controller: classical control architecture widely used by the industry for stabilization and regulation control.

The **extension** of PI control design to **infinite-dimensional systems** has attracted much attention in the recent years.

Early attempts:

- bounded control operators [Pohjolainen, 1982] [Pohjolainen, 1985];
- unbounded control operators [Xu and Jerbi, 1995].

PI regulation control of PDEs

State-of-the-art:

- PI boundary control of linear hyperbolic systems:
  [Bastin, Coron, and Tamasoiu, 2015]
  [Dos Santos, Bastin, Coron, and d'Andréa-Novel, 2008]
  [Lamare and Bekiaris-Liberis, 2015] [Xu and Sallet, 2014]
- PI boundary controller for 1-D nonlinear transport equation:
  [Trinh, Andrieu, and Xu, 2017] [Coron and Hayat, 2019]
- PI regulation control of drilling systems:
  [Barreau, Gouaisbaut, and Seuret, 2019]
  [Terrand-Jeanne, Martins, and Andrieu, 2018]
- Add of an integral component to open-loop exponentially stable semigroups: [Terrand-Jeanne, Andrieu, Martins, and Xu (2019)]

Objective: PI regulation control of a 1-D reaction-diffusion equation.

Problem setting

Let $L > 0$, let $c \in L^\infty(0, L)$, and let $D > 0$ be arbitrary.

$$\begin{aligned}
& y_t = y_{xx} + c(x)y + d(x), && (t,x) \in \mathbb{R}_+^* \times (0,L) \\
& y(t,0) = 0, && t \geq 0 \\
& y(t,L) = u_D(t) \triangleq u(t-D), && t \geq 0 \\
& y(0,x) = y_0(x), && x \in (0,L)
\end{aligned}$$

- $y(t,\cdot) \in L^2(0,L)$ is the state at time t;
- $u(t) \in \mathbb{R}$ is the control input;
- $D > 0$ is the (constant) control input delay;
- $d \in L^2(0,L)$ is a stationary distributed disturbance;
- $y_0 \in H^2(0,L)$ with $y_0(0) = 0$ and $y_0(L) = u(-D)$ is the initial condition.

Control design objective

Let $L > 0$, let $c \in L^{\infty}(0, L)$, and let $D > 0$ be arbitrary.

$$
\begin{aligned}
& y_t = y_{xx} + c(x)y + d(x), && (t,x) \in \mathbb{R}_+^* \times (0, L) \\
& y(t,0) = 0, && t \geq 0 \\
& y(t,L) = u_D(t) \triangleq u(t-D), && t \geq 0 \\
& y(0,x) = y_0(x), && x \in (0, L)
\end{aligned}
$$

Control design objective:

- Stabilization of the plant;
- PI regulation of the left Neumann trace $y_x(t,0)$ to some prescribed constant reference input $r \in \mathbb{R}$, i.e.,

$$y_x(t,0) \to r \quad \text{as} \quad t \to +\infty$$

- Regulation in spite of of the stationary distributed disturbance d;

4 PI regulation with delayed boundary control

Augmented system for PI feedback control

Add of the integral state $z(t)$

$$
\begin{aligned}
& y_t = y_{xx} + c(x)y + d(x), && (t,x) \in \mathbb{R}_+^* \times (0,L) \\
& \dot{z}(t) = y_x(t,0) - r, && t \geq 0 \\
& y(t,0) = 0, && t \geq 0 \\
& y(t,L) = u_D(t) \triangleq u(t-D), && t \geq 0 \\
& y(0,x) = y_0(x), && x \in (0,L) \\
& z(0) = z_0
\end{aligned}
$$

The system is uncontrolled for negative times, i.e. $u(t) = 0$ for $t < 0$.

We assume that $y_0 \in H^2(0,L) \cap H_0^1(0,L)$.

Equivalent homogeneous Dirichlet problem

The change of variable

$$w(t,x) = y(t,x) - \frac{x}{L} u_D(t)$$

yields the equivalent homogeneous Dirichlet problem:

$$\begin{aligned}
w_t &= w_{xx} + c(x)w + \frac{x}{L}c(x)u_D - \frac{x}{L}\dot{u}_D + d(x) \\
\dot{z}(t) &= w_x(t,0) + \frac{1}{L}u_D(t) - r \\
w(t,0) &= w(t,L) = 0 \\
w(0,x) &= y_0(x) - \frac{x}{L}u_D(0) = y_0(x) \\
z(0) &= z_0
\end{aligned}$$

Abstract formulation of the problem

Introducing the operator $\mathcal{A} = \partial_{xx} + c\,\mathrm{id} : D(\mathcal{A}) \subset L^2(0,L) \rightarrow L^2(0,L)$ defined on the domain $D(\mathcal{A}) = H^2(0,L) \cap H_0^1(0,L)$,

$$w_t(t,\cdot) = \mathcal{A}w(t,\cdot) + a(\cdot)u_D(t) + b(\cdot)\dot{u}_D(t) + d(\cdot)$$
$$\dot{z}(t) = w_x(t,0) + \frac{1}{L}u_D(t) - r$$

with $a(x) = \frac{x}{L}c(x)$ and $b(x) = -\frac{x}{L}$.

Key properties: $\mathcal{A}$ is self-adjoint, has compact resolvent, and has simple eigenvalues. Hence we have a Hilbert basis $(e_j)_{j\geq 1}$ of $L^2(0,L)$ consisting of eigenfunctions of $\mathcal{A}$ associated with the sequence of simple real eigenvalues

$$-\infty < \cdots < \lambda_j < \cdots < \lambda_1$$

with (when $j \rightarrow +\infty$)

$$e_j'(0) \sim \sqrt{\frac{2}{L}}\sqrt{|\lambda_j|}, \qquad \lambda_j \sim -\frac{\pi^2 j^2}{L^2}$$

Spectral reduction of the problem

Since $w(0,\cdot) = y_0 \in H^2(0,L) \cap H_0^1(0,L)$, the classical solution $w(t,\cdot) \in H^2(0,L) \cap H_0^1(0,L)$ can be expanded as a series in the eigenfunctions $e_j(\cdot)$, convergent in $H_0^1(0,L)$,

$$w(t,\cdot) = \sum_{j=1}^{+\infty} w_j(t) e_j(\cdot).$$

Equivalent infinite-dimensional control system:

$$\begin{aligned} \dot{w}_j(t) &= \lambda_j w_j(t) + a_j u_D(t) + b_j \dot{u}_D(t) + d_j \\ \dot{z}(t) &= \sum_{j \geq 1} w_j(t) e_j'(0) + \frac{1}{L} u_D(t) - r \end{aligned}$$

for $j \in \mathbb{N}^*$, with $w_j(t) = \langle w(t,\cdot), e_j \rangle$, $a_j = \langle a, e_j \rangle$, $b_j = \langle b, e_j \rangle$, and $d_j = \langle d, e_j \rangle$.

Auxiliary control input $v = \dot{u}$

Introducing the auxiliary control input $v = \dot{u}$, and denoting $v_D(t) \triangleq v(t-D)$,

$$\begin{aligned} \dot{u}_D(t) &= v_D(t) \\ \dot{w}_j(t) &= \lambda_j w_j(t) + a_j u_D(t) + b_j v_D(t) + d_j \\ \dot{z}(t) &= \sum_{j \geq 1} w_j(t) e_j'(0) + \frac{1}{L} u_D(t) - r \end{aligned}$$

for $j \in \mathbb{N}^*$.

As $u(t) = 0$ for $t < 0$, we also have $v(t) = 0$ for $t < 0$ and the initial condition $u_D(0) = 0$.

Finite-dimensional truncated model

Let $N_0 \in \mathbb{N}^*$ be such that $\lambda_j \geq 0$ when $1 \leq j \leq N_0$ and $\lambda_j \leq \lambda_{N_0+1} < 0$ when $j \geq N_0 + 1$. Introducing:

$$X_1(t) = \begin{pmatrix} u_D(t) \\ w_1(t) \\ \vdots \\ w_{N_0}(t) \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_1 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \cdots & \lambda_{N_0} \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 1 & b_1 & \dots & b_{N_0} \end{pmatrix}^\top,$$
$$D_1 = \begin{pmatrix} 0 & d_1 & \dots & d_{N_0} \end{pmatrix}^\top,$$

the N_0 first modes of the PDE are captured by

$$\dot{X}_1(t) = A_1 X_1(t) + B_1 v_D(t) + D_1.$$

Rewriting of the integral component

Integral component:

$$\dot{z}(t) = \sum_{j=1}^{N_0} w_j(t) e_j'(0) + \sum_{j \geq N_0+1} w_j(t) e_j'(0) + \frac{1}{L} u_D(t) - r.$$

Change of variable (recall that $\left| \frac{e_j'(0)}{\lambda_j} \right|^2 \sim \frac{2L}{\pi^2 j^2}$ when $j \to +\infty$):

$$\zeta(t) \triangleq z(t) - \sum_{j \geq N_0+1} \frac{e_j'(0)}{\lambda_j} w_j(t),$$

whose time derivative is given by

$$\dot{\zeta}(t) = \alpha u_D(t) + \beta v_D(t) - \gamma + \sum_{j=1}^{N_0} w_j(t) e_j'(0),$$

with

$$\alpha = \frac{1}{L} - \sum_{j \geq N_0+1} \frac{e_j'(0)}{\lambda_j} a_j, \quad \beta = - \sum_{j \geq N_0+1} \frac{e_j'(0)}{\lambda_j} b_j, \quad , \gamma = r + \sum_{j \geq N_0+1} \frac{e_j'(0)}{\lambda_j} d_j.$$

Augmented truncated model

With $X(t) = \begin{bmatrix} X_1(t)^\top & \zeta(t) \end{bmatrix}^\top \in \mathbb{R}^{N_0+2}$ and the exogenous input $\Gamma = \begin{bmatrix} D_1^\top & -\gamma \end{bmatrix}^\top \in \mathbb{R}^{N_0+2}$,

$$\dot{X}(t) = AX(t) + Bv(t-D) + \Gamma$$

where

$$A = \begin{pmatrix} A_1 & 0 \\ L_1 & 0 \end{pmatrix} \in \mathbb{R}^{(N_0+2)\times(N_0+2)}, \quad B = \begin{pmatrix} B_1 \\ \beta \end{pmatrix} \in \mathbb{R}^{N_0+2},$$

with

$$L_1 = \begin{pmatrix} \alpha & e_1'(0) & \dots & e_{N_0}'(0) \end{pmatrix} \in \mathbb{R}^{1\times(N_0+1)}.$$

Representation for control design and stability analysis

Final representation of the reaction-diffusion equation augmented with the integral component:

$$\dot{X}(t) = AX(t) + Bv(t-D) + \Gamma$$
$$\dot{w}_j(t) = \lambda_j w_j(t) + a_j u(t-D) + b_j v_D(t) + d_j$$

with $j \geq N_0 + 1$.

Lemma

The pair (A, B) satisfies the Kalman condition.

Control strategy

Design of a classical **predictor feedback** to stabilize the truncated model:

$$\dot{X}(t) = AX(t) + Bv(t-D) + \Gamma.$$

Introducing the Artstein transformation [Artstein, 1982]

$$Z(t) = X(t) + \int_{t-D}^{t} e^{A(t-D-\tau)} Bv(\tau) \, \mathrm{d}\tau,$$

we have

$$\dot{Z}(t) = AZ(t) + e^{-DA} Bv(t) + \Gamma.$$

Let $K \in \mathbb{R}^{1\times(N_0+2)}$ be such that $A_K = A + e^{-DA}BK$ is Hurwitz. Setting $v(t) = \chi_{[0,+\infty)}(t)KZ(t)$, we obtain the stable closed-loop dynamics

$$\dot{Z}(t) = A_K Z(t) + \Gamma.$$

System in closed-loop

Closed-loop dynamics in X-coordinates:

$$\begin{aligned}\dot{X}(t) &= AX(t) + Bv_D(t) + \Gamma \\ \dot{w}_j(t) &= \lambda_j w_j(t) + a_j u_D(t) + b_j v_D(t) + d_j, \quad j \geq N_0 + 1 \\ v(t) &= \chi_{[0,+\infty)}(t) K \left(X(t) + \int_{\max(t-D,0)}^{t} e^{A(t-D-\tau)} B v(\tau) \, \mathrm{d}\tau \right)\end{aligned}$$

Closed-loop dynamics in Z-coordinates:

$$\begin{aligned}\dot{Z}(t) &= A_K Z(t) + \Gamma \\ \dot{w}_j(t) &= \lambda_j w_j(t) + a_j u_D(t) + b_j v_D(t) + d_j, \quad j \geq N_0 + 1 \\ v(t) &= \chi_{[0,+\infty)}(t) K Z(t)\end{aligned}$$

Dynamics of deviations

The **equilibrium condition** of the closed-loop system is fully characterized by:

- the constant reference input r for the left Neumann trace $y_x(t,0)$;
- the stationary distributed disturbance $d \in L^2(0,L)$.

Dynamics of **deviations** in X-coordinates:

$$\Delta \dot{X}(t) = A\Delta X(t) + B\Delta v_D(t)$$

$$\Delta \dot{w}_j(t) = \lambda_j \Delta w_j(t) + a_j \Delta u_D(t) + b_j \Delta v_D(t), \quad j \geq N_0 + 1$$

$$\Delta v(t) = \chi_{[0,+\infty)}(t) K \left(\Delta X(t) + \int_{\max(t-D,0)}^{t} e^{A(t-D-\tau)} B \Delta v(\tau) \, \mathrm{d}\tau \right)$$

Similar result for the dynamics of deviations in Z-coordinates.

4 PI regulation with delayed boundary control

- Control design strategy
- **Stability analysis**
- Numerical application
- Extensions

Main stability result

Theorem (stability) [Lhachemi, Prieur, and Trélat, 2020]

There exist $\kappa, \overline{C}_1 > 0$ such that

$$\Delta u_D(t)^2 + \Delta\zeta(t)^2 + \|\Delta w(t)\|^2_{H^1_0(0,L)} \leq \overline{C}_1 e^{-2\kappa t}\left(\Delta u_D(0)^2 + \Delta\zeta(0)^2 + \|\Delta w(0)\|^2_{H^1_0(0,L)}\right), \quad \forall t \geq 0.$$

The proof of the Theorem relies on the following Lyapunov function:

$$V(t) = \frac{M}{2}\Delta Z(t)^\top P \Delta Z(t) + \frac{M}{2}\int_{\max(t-D,0)}^{t} \Delta Z(s)^\top P \Delta Z(s)\,\mathrm{d}s - \frac{1}{2}\sum_{j\geq 1}\lambda_j \Delta w_j(t)^2,$$

where $P = P^\top \in \mathbb{R}^{(N_0+2)\times(N_0+2)}$ is the solution of the Lyapunov equation $A_K^\top P + PA_K = -I$ and $M > 0$ is a constant chosen sufficiently large.

Sketch of proof

Lemma 1

There exists a constant $C_1 > 0$ such that

$$V(t) \geq C_1 \sum_{j \geq 1} (1 + |\lambda_j|) \Delta w_j(t)^2, \quad \forall t \geq 0$$

$$V(t) \geq C_1 \left(\Delta u_D(t)^2 + \Delta\zeta(t)^2 + \|\Delta w(t)\|^2_{H_0^1(0,L)} \right), \quad \forall t \geq 0$$

$$V(t) \geq C_1 \|\Delta Z(t)\|^2, \quad \forall t \geq 0.$$

Lemma 2

There exist $\kappa > 0$ such that

$$V(t) \leq e^{-2\kappa(t-D)} V(D), \quad \forall t \geq D.$$

Lemma 3

There exists $C_2 > 0$ such that

$$V(t) \leq C_2 \left(\Delta u_D(0)^2 + \Delta\zeta(0)^2 + \|\Delta w(0)\|^2_{H_0^1(0,L)} \right), \quad \forall t \in [0, D].$$

Assessment of the reference tracking

Theorem (reference tracking) [Lhachemi, Prieur, and Trélat, 2020]

Let $\kappa > 0$ be provided by the previous stability Theorem. There exists $\overline{C}_2 > 0$ such that

$$
\begin{aligned}
&|y_x(t,0) - r| \\
&\leq \overline{C}_2 e^{-\kappa t} \left(|\Delta u_D(0)| + |\Delta \zeta(0)| + \|\Delta w(0)\|_{H_0^1(0,L)} + \|\mathcal{A} \Delta w(0)\|_{L^2(0,L)} \right).
\end{aligned}
$$

Sketch of proof

Since $w_{e,x}(0) + \frac{1}{L} u_e = r$, we have

$$|y_x(t,0) - r| = \left| w_x(t,0) + \frac{1}{L} u_D(t) - r \right|$$
$$\leq |w_x(t,0) - w_{e,x}(0)| + \frac{1}{L} |\Delta u_D(t)|.$$

As $e_j'(0) \sim \sqrt{\frac{2}{L}} \sqrt{|\lambda_j|}$, there exists a constant $\gamma_7 > 0$ such that $|e_j'(0)| \leq \gamma_7 \sqrt{|\lambda_j|}$ for all $j \geq N_0 + 1$. For any $m \geq N_0 + 1$,

$$|w_x(t,0) - w_{e,x}(0)|$$
$$\leq \sum_{j=1}^{m-1} |\Delta w_j(t)| |e_j'(0)| + \gamma_7 \sum_{j \geq m} \sqrt{|\lambda_j|} |\Delta w_j(t)|$$
$$\leq \sqrt{\sum_{j=1}^{m-1} e_j'(0)^2} \sqrt{\sum_{j=1}^{m-1} \Delta w_j(t)^2} + \gamma_7 \sqrt{\sum_{j \geq m} \frac{1}{|\lambda_j|}} \sqrt{\sum_{j \geq m} \lambda_j^2 \Delta w_j(t)^2}$$

Sketch of proof

It remains to study the term $\sqrt{\sum_{j \geq m} \lambda_j^2 \Delta w_j(t)^2}$. Recall that

$$\Delta \dot{w}_j(t) = \lambda_j \Delta w_j(t) + a_j \Delta u_D(t) + b_j \Delta v_D(t).$$

Hence, by direct integration ($j \geq m \geq N_0 + 1$)

$$\begin{aligned} & |\lambda_j \Delta w_j(t)| \\ & \leq e^{\lambda_j t} |\lambda_j \Delta w_j(0)| + \int_0^t (-\lambda_j) e^{\lambda_j (t-\tau)} \left\{ |a_j| |\Delta u_D(\tau)| + |b_j| |\Delta v_D(\tau)| \right\} \mathrm{d}\tau \end{aligned}$$

Using the previous stability result, we obtain

$$\begin{aligned} & \sum_{j \geq m} \lambda_j^2 \Delta w_j(t)^2 \\ & \leq C_3^2 e^{-2\kappa t} \left(|\Delta u_D(0)|^2 + |\Delta \zeta(0)|^2 + \|\Delta w(0)\|_{H_0^1(0,L)}^2 + \|\mathcal{A} \Delta w(0)\|_{L^2(0,L)}^2 \right) \end{aligned}$$

for some constant $C_3 > 0$.

Numerical application

$$\begin{aligned} & y_t = y_{xx} + c(x)y + d(x), && (t,x) \in \mathbb{R}_+^* \times (0,L) \\ & y(t,0) = 0, && t \geq 0 \\ & y(t,L) = u(t-D), && t \geq 0 \\ & y(0,x) = y_0(x), && x \in (0,L) \end{aligned}$$

Numerical setting:

- system parameters: $c = 1.25$, $L = 2\pi$, and $D = 1\,\mathrm{s}$;
- first eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 0.25$, $\lambda_3 = -1$;
- control design: $N_0 = 2$, gain $K \in \mathbb{R}^{1\times 4}$ is computed to place the poles of the closed-loop truncated model at -0.5, -0.6, -0.7, and -0.8;
- reference: $r = 50$;
- distributed disturbance: $d(x) = x$;
- initial condition: $y_0(x) = -\frac{x}{L}\left(1 - \frac{x}{L}\right)$;

Numerical application

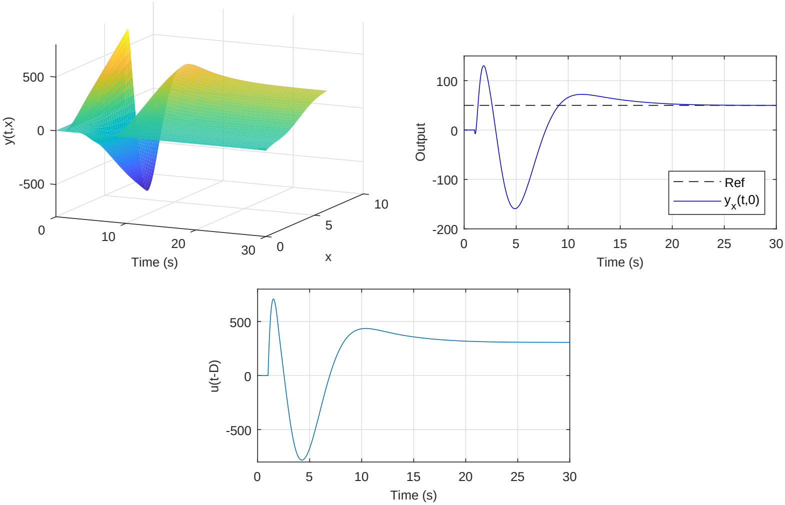


Figure: Time evolution of the closed-loop system

4. PI regulation with delayed boundary control
   - Control design strategy
   - Stability analysis
   - Numerical application
   - Extensions

Extension 1: time-varying case

Let $L > 0$, let $c \in L^\infty(0, L)$, and let $D > 0$ be arbitrary.

$$\begin{aligned} &y_t = y_{xx} + c(x)y + d(t, x), && (t, x) \in \mathbb{R}_+^* \times (0, L) \\ &y(t, 0) = 0, && t \geq 0 \\ &y(t, L) = u(t - D), && t \geq 0 \\ &y(0, x) = y_0(x), && x \in (0, L) \end{aligned}$$

PI control:

- exponential input-to-state stabilization w.r.t. $d(t, x)$;
- setpoint regulation of the left Neumann trace $y_x(t, 0)$ to some reference input $r(t) \in \mathbb{R}$.

[Lhachemi, Prieur, and Trélat, 2021]

Extension 2: semilinear wave equation

$$
\begin{aligned}
&y_{tt} = y_{xx} + f(y), && (t,x) \in \mathbb{R}_+^* \times (0,L) \\
&y(t,0) = 0, && t \geq 0 \\
&y_x(t,L) = u(t), && t \geq 0 \\
&y(0,x) = y_0(x), && x \in (0,L) \\
&y_t(0,x) = y_1(x), && x \in (0,L)
\end{aligned}
$$

Control strategy:

1. preliminary (classical) velocity feedback;
2. spectral reduction-based design of a PI controller.

Result: Local PI regulation control of the left Neumann trace $y_x(t,0)$ to some prescribed constant reference $r \in \mathbb{R}$.

[Lhachemi, Prieur, and Trélat, 2020]

Conclusion

- Boundary stabilization and regulation control of PDEs in the presence of delays.
- Spectral reduction-based methods can be efficient tools to achieve:
  - stabilization with delayed boundary control;
  - boundary stabilization in the presence of a state-delay;
  - PI regulation control.
- Future lines of research:
  - robustness;
  - output feedback;
  - systems of PDEs;
  - etc.

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