Input-to-state stability of time-delay systems: Lyapunov-Krasovskii characterizations and feedback control redesign

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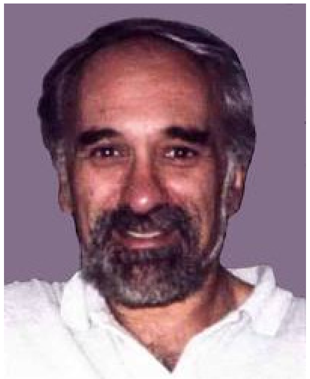
Outline

- ISS, ISS-ation for Delay-Free Systems
- ISS for Systems Described by RFDEs,
- ISS for Systems Described by F$\mathcal{D}$Es
- ISS for Systems Described by NFDEs
- ISS-ation of Systems Described by RFDEs
- A Case Study: the Chemical Reactor with Recycle
- Conclusions

 E. D. Sontag,

Northeastern University, Boston, Massachusetts, USA

E.D. Sontag, Smooth stabilization implies coprime factorization, *IEEE Transactions on Automatic Control,* Vol. 34, No. 4, pp. 435–443, 1989.

A function $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ is:

- positive definite if it is continuous, zero at zero and $\delta(s) > 0$ for all $s > 0$ (ex: $s \to \frac{s}{1+s^2}$);

- of class $\mathcal{K}$ if it is positive definite and strictly increasing (ex: $s \to 1 - e^{-s}$);

- of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and it is unbounded

  (ex: $s \to s^2$);

- of class $\mathcal{L}$ if it is continuous and it monotonically decreases to zero as its argument tends to $+\infty$ (ex: $s \to e^{-s}$).

A function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each $t \geq 0$ and $\beta(s, \cdot)$ is of class $\mathcal{L}$ for each $s \geq 0$

(ex: $(s, t) \to se^{-t}$).

For positive real Δ, positive integer n, $C([-\Delta, 0]; R^n)$ denotes the Banach space of the continuous functions mapping $[-\Delta, 0]$ into R^n, endowed with the supremum norm, denoted with the symbol $\| \cdot \|_\infty$.

The symbol $\| \cdot \|_a$ denotes any semi-norm in $C([-\Delta, 0]; R^n)$ for which there exist two positive reals $\underline{\gamma}_a$ and $\overline{\gamma}_a$ such that, for any $\phi \in C([-\Delta, 0]; R^n)$, the following inequalities hold

$$\underline{\gamma}_a |\phi(0)| \leq \|\phi\|_a \leq \overline{\gamma}_a \|\phi\|_\infty$$

A functional $V : C([-\Delta, 0]; R^n) \to R^+$ is Fréchet differentiable at a point $\phi \in C([-\Delta, 0]; R^n)$, if there exists a linear bounded operator, which is called the Fréchet differential at ϕ and is denoted as $D_F V(\phi)$, mapping $C([-\Delta, 0]; R^n)$ into R, such that

$$\lim_{\psi \to \mathbf{0}} \frac{|V(\phi + \psi) - V(\phi) - D_F V(\phi)\psi|}{\|\psi\|_\infty} = 0$$

In the following:

- RFDE stands for Retarded Functional Differential Equation.
- NFDE stands for Neutral Functional Differential Equation.
- F$\mathcal{D}$E stands for Functional Difference Equation.
- ISS stands for Input-to-State Stability, or Input-to-State Stable.

ISS Definition (Sontag, 1989)

$$\dot{x}(t) = f(x(t), v(t)), \ a.e. \qquad x(t) \in R^n, \ v(t) \in R^m, \qquad x(0) = x_0 \tag{1}$$

(f locally Lipschitz)

Definition 1. *The system described by (1) is ISS if there esixt $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that, for any initial state x_0 and any Lebesgue measurable and locally essentially bounded input v, the solution exists for all $t \geq 0$ and, furthermore, satisfies the inequality*

$$|x(t)| \leq \beta(|x_0|, t) + \gamma(\|v_{[0,t)}\|_\infty), \quad t \geq 0$$

Liapunov Characterization of ISS

Sontag & Wang, SCL, 1995, Lin, Sontag, Wang, SICON, 1996

Theorem 2. *The system described by the ODE (1) is ISS if and only if there exist a smooth function* $V : R^n \to R^+$*, functions* α_1*,* α_2*,* α_3 *of class* $\mathcal{K}_\infty$*, function* ρ *of class* $\mathcal{K}$*, such that*

$H_1)$ $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$, $\forall x \in R^n$;

$H_2)$ $\frac{\partial V(x)}{\partial x} f(x, v) \leq -\alpha_3(|x|) + \rho(|v|)$, $\forall x \in R^n, v \in R^m$

ISS-ation (Sontag, 1989)

$\dot{x}(t) = f(x(t)) + g(x(t))(u(t) + d(t))$

Hp) $u(t) = k(x(t))$ is stabilizing when $d \equiv 0$, $V : R^n \to R^+$ is a Liapunov function for $\dot{x}(t) = f(x(t)) + g(x(t))k(x(t))$, i.e.: $\alpha_1(|x|) \le V(x) \le \alpha_2(|x|)$, $\frac{\partial V(x)}{\partial x}(f(x) + g(x)k(x)) \le -\alpha_3(|x|)$;

Th) $u_s(t) = k(x(t)) - \left[\frac{\partial V(x(t))}{\partial x(t)} g(x(t))\right]^T$ is ISS-ing, i.e.

$$\dot{x}(t) = f(x(t)) + g(x(t))(u_s(t) + d(t))$$

is ISS w.r.t. the disturbance $d(t)$.

Example (Sontag, 1989)

$\dot{x}(t) = x(t) + (1 + x^2(t))(u(t) + d(t))$

If $d(t) \equiv 0$, then $u(t) = -\frac{2x(t)}{1+x^2(t)}$ is a stabilizing feedback control law. Indeed, the closed-loop system becomes $\dot{x}(t) = -x(t)$.

But, with this feedback control law, the closed-loop system is described, in the case $d(t) \neq 0$, by the equation

$$\dot{x}(t) = -x(t) + (1 + x^2(t))d(t),$$

and it can easy become unstable, for instance by suitable constant disturbance $d(t)$.

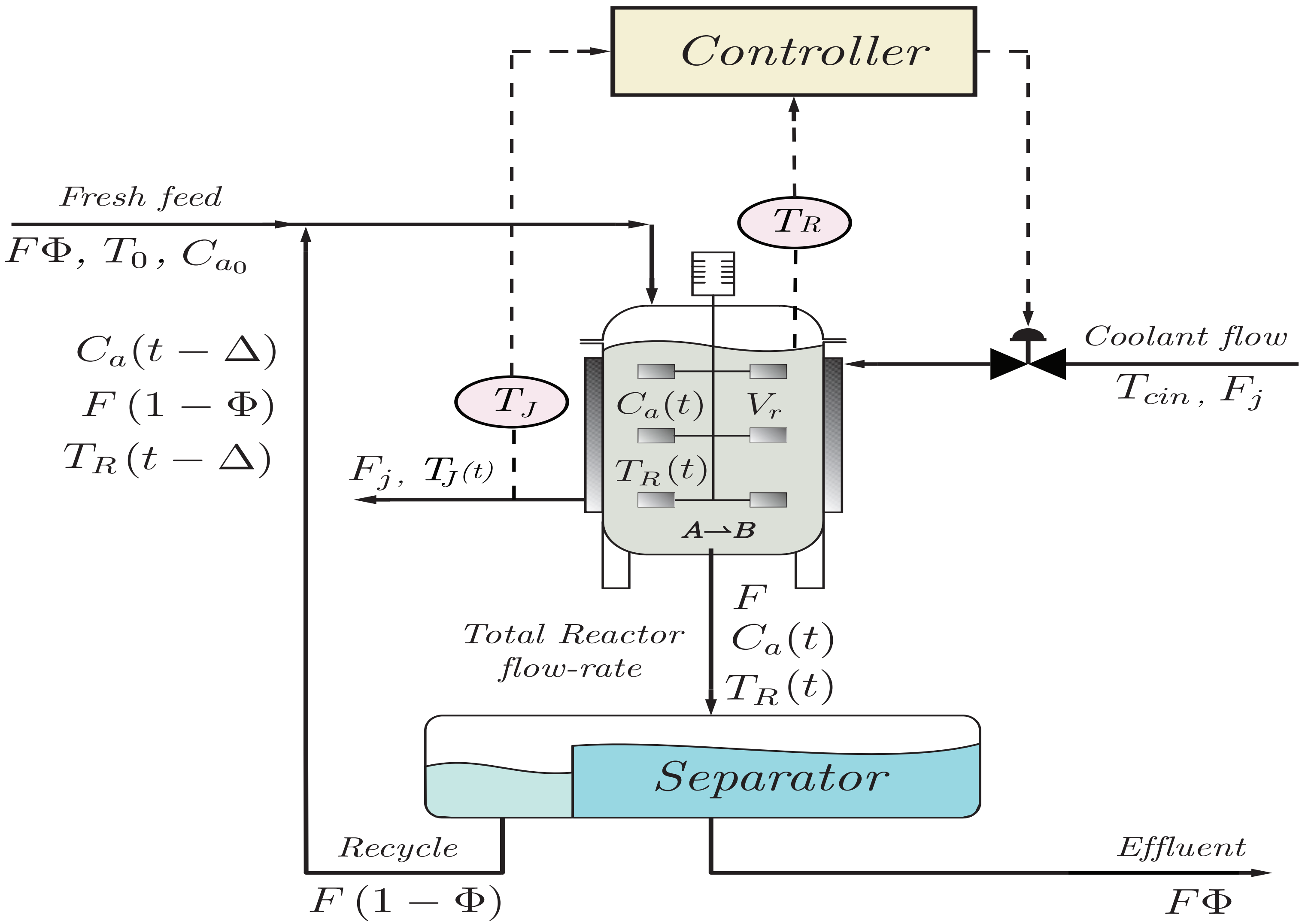
Now, we consider a Liapunov function for the disturbance-free closed loop system $\dot{x}(t) = -x(t)$. We can choose $V(x) = x^2$. Then we have the new feedback control law

$$u_s(t) = -\frac{2x(t)}{1 + x^2(t)} - 2x(t)\left(1 + x^2(t)\right)$$

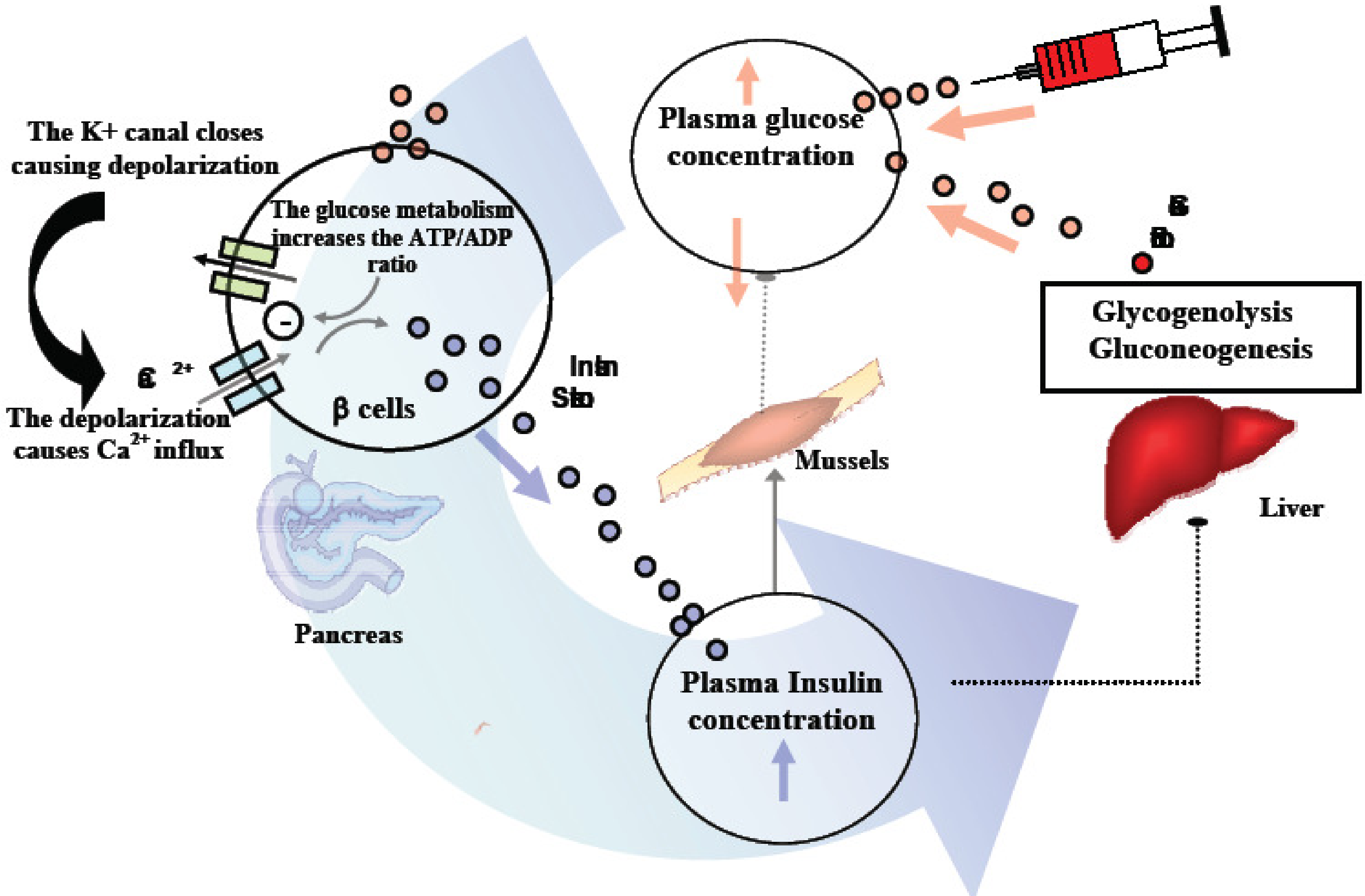
The new closed-loop system becomes

$$\dot{x}(t) = -x(t) - 2x(t)\left(1 + x^2(t)\right)^2 + \left(1 + x^2(t)\right) d(t)$$

This system is ISS w.r.t. the disturbance $d(t)$.



Continuous Stirred Tank Reactor. Delays appear because of the recycle.



Human Glucose-Insulin System. Delays occur because of the reaction time of the pancreas to plasma-glucose variations.

The beginning of ISS for time-delay systems

A.R. Teel, Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem, *IEEE Transactions on Automatic Control*, Vol. 43, No. 7, pp. 960–964, 1998.

P. Pepe, and Z.-P. Jiang, A Lyapunov-Krasovskii methodology for ISS and iISS of time-delay systems, *Systems & Control Letters*, Vol. 55, No. 12, pp. 1006–1014, 2006.

E. Fridman, M. Dambrine, N. Yeganefar, On input-to-state stability of systems with time-delay: A matrix inequalities approach, Automatica, Vol. 44, N. 9, pp. 2364-2369, 2008.

Systems Described by RFDEs

$$\dot{x}(t) = f(x_t, v(t)), \qquad t \geq 0, \qquad a.e.,$$
$$x(\tau) = x_0(\tau), \qquad \tau \in [-\Delta, 0], \tag{2}$$

$f : C([-\Delta, 0]; R^n) \times R^m \to R^n$ Lipschitz on bounded sets,

$$x_t \in C([-\Delta, 0]; R^n), \qquad x_t(\tau) = x(t + \tau),\ \tau \in [-\Delta, 0]$$

An example (recall $x_t(\tau) = x(t+\tau)$, $\tau \in [-\Delta, 0]$):

$$\dot{x}(t) = x^4(t) + x^3(t-\pi) + x^2(t-e) + x\left(t - \sqrt{3}\right) + \int_{t-\sqrt{2}}^{t} x^5(s)ds + v(t) \tag{3}$$

Setting $\Delta = \pi$ (maximum involved time delay), by equalities

$$x(t) = x_t(0),\ x(t-\pi) = x_t(-\pi),\ x(t-e) = x_t(-e),\ x\left(t-\sqrt{3}\right) = x_t\left(-\sqrt{3}\right),$$

$$\int_{t-\sqrt{2}}^{t} x^5(s)ds = \int_{-\sqrt{2}}^{0} x^5(t+\tau)d\tau = \int_{-\sqrt{2}}^{0} x_t^5(\tau)d\tau,$$

the system described by (3) can be rewritten in the form $\dot{x}(t) = f(x_t, v(t))$, where $f : C([-\Delta, 0]; R) \times R \to R$ is defined, for $\phi \in C([-\Delta, 0]; R)$, $u \in R$, as

$$f(\phi, u) = \phi^4(0) + \phi^3(-\pi) + \phi^2(-e) + \phi\left(-\sqrt{3}\right) + \int_{-\sqrt{2}}^{0} \phi^5(s)ds + u$$

Existence and Uniqueness of the Solution

Theorem 3. *For any initial condition* $x_0 \in C([-\Delta, 0]; R^n)$ *and any Lebesgue measurable and locally essentially bounded input function* u*, the RFDE (2) admits a unique locally absolutely continuous solution* $x(t)$ *on a maximal time interval* $[0, b)$, $0 < b \leq +\infty$*. If* $b < +\infty$*, then the solution is unbounded in* $[0, b)$.

Stability Definitions

Definition 4. *Let in the RFDE (2)* $u(t) \equiv 0$*. The system described by the RFDE (2) is said to be* $0-$*GAS if there exist a function* β *of class* $\mathcal{KL}$ *such that, for any initial state* $x_0 \in C([-\Delta, 0]; R^n)$*, the corresponding solution exists for all* $t \geq 0$ *and, furthermore, satisfies the inequality*

$$|x(t)| \leq \beta(\|x_0\|_\infty, t), \;\; \forall t \geq 0 \tag{4}$$

Definition 5. *The system described by the RFDE (2) is said to be ISS if there exist a function* β *of class* $\mathcal{KL}$ *and a function* γ *of class* $\mathcal{K}$ *such that, for any initial state* $x_0 \in C([-\Delta, 0]; R^n)$ *and any Lebesgue measurable, locally essentially bounded input* v*, the corresponding solution exists for all* $t \geq 0$ *and, furthermore, satisfies*

$$|x(t)| \leq \beta(\|x_0\|_\infty, t) + \gamma(\|v_{[0,t)}\|_\infty), \qquad \forall t \geq 0.$$

Definition 6. *Let* $V : C([-\Delta, 0]; R^n) \to R^+$ *be a continuous functional. The derivative* $D^+V : C([-\Delta, 0]; R^n) \times R^m \to R^\star$ *of the functional* V *is defined, in the Driver's form (see Driver, 1962, Burton, 1985, Pepe & Jiang, 2006, Karafyllis, 2006), for* $\phi \in C([-\Delta, 0]; R^n)$, $v \in R^m$, *as follows*

$$D^+V(\phi, v) = \limsup_{h \to 0^+} \frac{1}{h} \left(V\left(\phi_{h,v}\right) - V(\phi) \right), \tag{5}$$

where, for $h \in [0, \Delta)$, $\phi_{h,v} \in C([-\Delta, 0]; R^n)$ *is given by*

$$\phi_{h,v}(s) = \begin{cases} \phi(s+h), & s \in [-\Delta, -h), \\ \phi(0) + f(\phi, v)(h+s), & s \in [-h, 0] \end{cases} \tag{6}$$

Theorem 7. *Let in the RFDE (2)* $u(t) = 0$, $t \geq 0$. *The system described by the RFDE (2) is* 0−*GAS if and only if there exist a locally Lipschitz functional* $V : C([-\Delta, 0]; R^n) \to R^+$ *and functions* α_1, α_2 *of class* $\mathcal{K}_\infty$, α_3 *of class* $\mathcal{K}$, *such that,* $\forall \phi \in C([-\Delta, 0]; R^n)$, *the following inequalities hold:*

i) $\alpha_1(|\phi(0)|) \leq V(\phi) \leq \alpha_2(\|\phi\|_\infty)$*;*

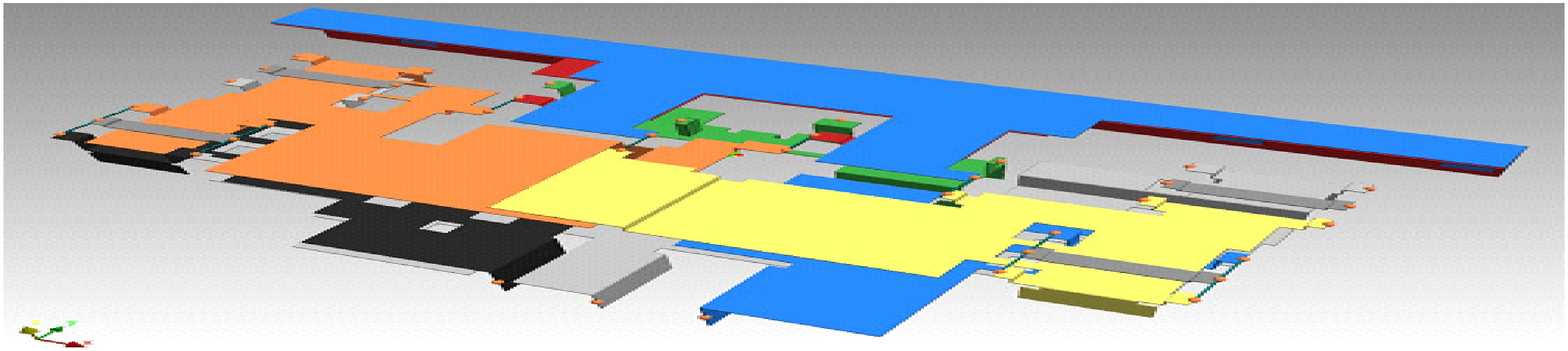
ii) $D^+V(\phi, 0) \leq -a_3(|\phi(0)|)$

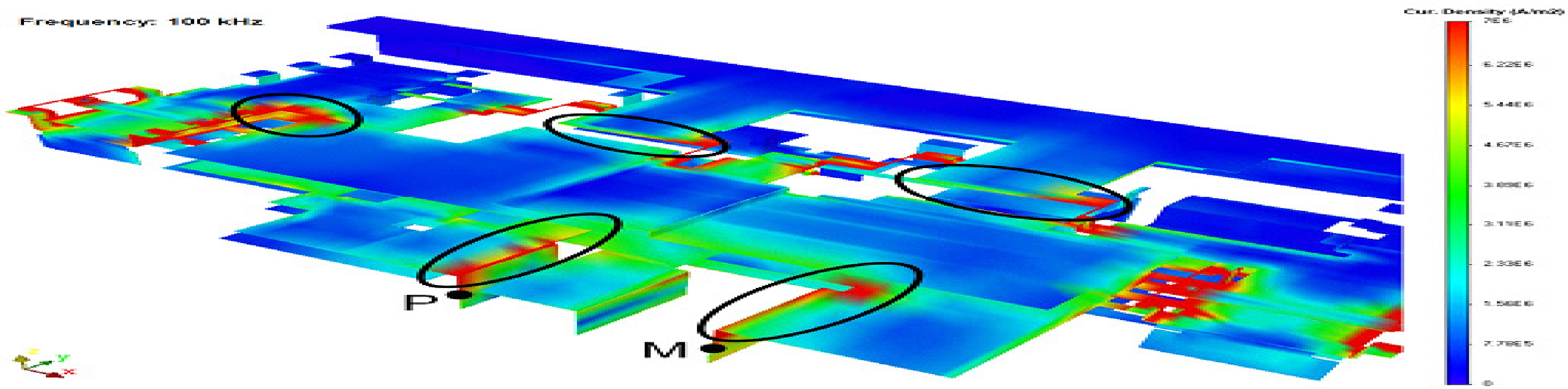
Theorem 8. *(Karafyllis, Pepe & Jiang, 2006, 2008) The system described by the RFDE (2) is ISS if and only if there exist a locally Lipschitz functional* $V : C([-\Delta, 0]; R^n) \to R^+$*, a semi-norm* $\|\cdot\|_a$ *in* $C([-\Delta, 0]; R^n)$*, functions* α_1, α_2*,* α_3 *of class* $\mathcal{K}_\infty$*, a function* ρ *of class* $\mathcal{K}$ *such that:*

i) $\alpha_1(|\phi(0)|) \leq V(\phi) \leq \alpha_2(\|\phi\|_a), \ \forall \phi \in \mathcal{C}$*;*

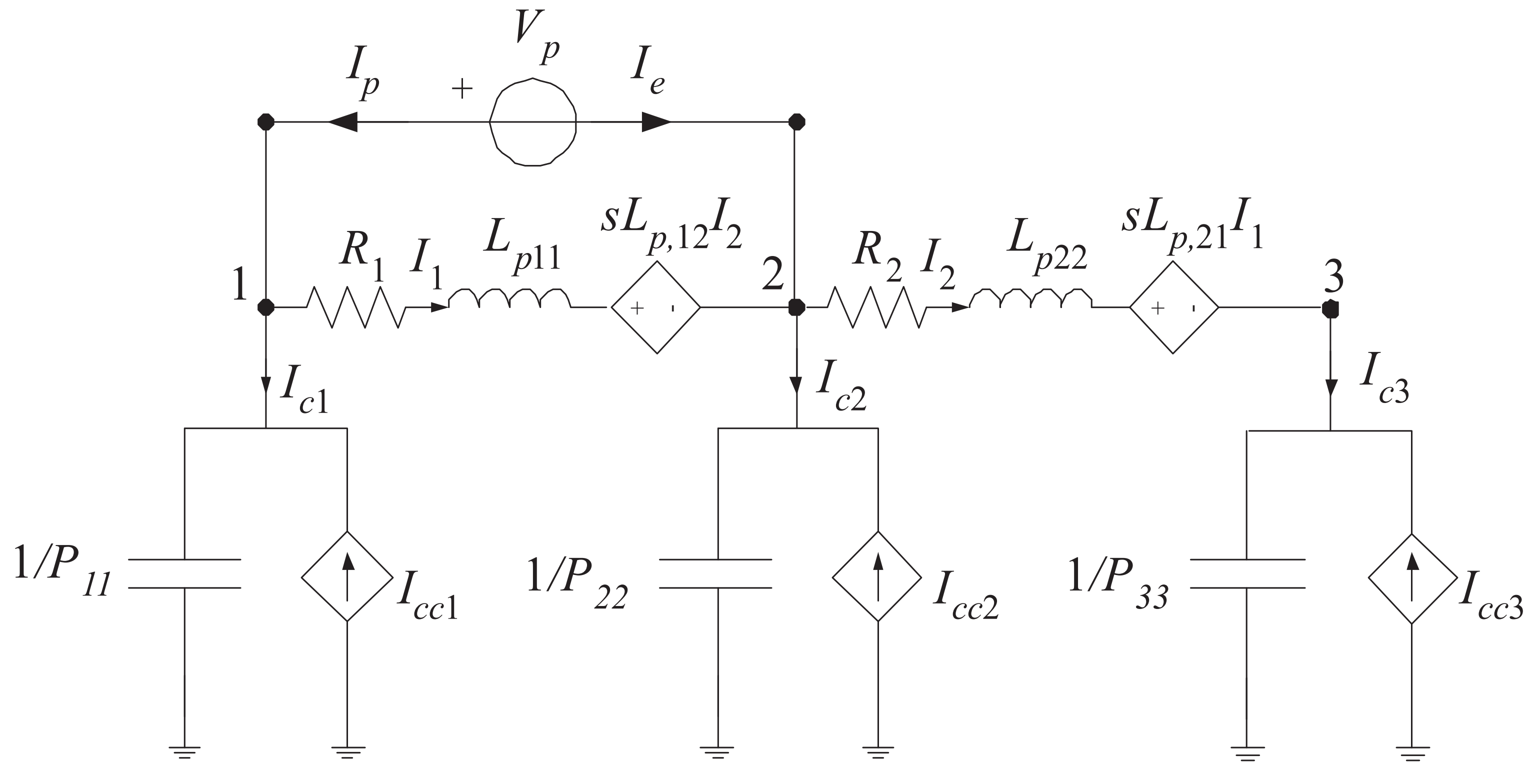
ii) $D^+V(\phi, d) \leq -\alpha_3(\|\phi\|_a) + \rho(|d|), \ \ \forall \ \phi \in \mathcal{C}, \ d \in R^m$

- Recall that $\underline{\gamma}_a|\phi(0)| \leq \|\phi\|_a \leq \overline{\gamma}_a\|\phi\|_\infty$





Example of Copper Interconnections System for a Converter. More Red Regions Correspond to Higher Currents. Modelled by Partial Element Equivalent Circuits (PEECs).



Partial Element Equivalent Circuits (here an example is reported) describe electromagnetic problems, they are a circuit interpretation of the Maxwell Equations, when the space is suitably discretized. The electric and magnetic interactions do happen at distances and with propagation times, since the electromagnetic field propagates, at most, at the light speed. Thus delays are involved, which, in a state space description, affect both the state and its derivative (neutral-type systems). See papers by A. Bellen, N. Guglielmi, A. Ruehli, G. Antonini, X.-M. Zhang, Q.-L. Han, P. Pepe.

NFDEs in Hale's Form

$$\frac{d}{dt}(\mathcal{D}x_t) = f(x_t, v(t)), \qquad t \geq 0,\ a.e.,$$
$$x(\tau) = x_0(\tau),\ \tau \in [-\Delta, 0],\ x_0 \in C([-\Delta, 0]; R^n) \tag{7}$$

where: $x(t) \in R^n$; $v(t) \in R^m$ is the input, measurable and locally essentially bounded, n, m are positive integers; $\mathcal{D} : C([-\Delta, 0]; R^n) \to R^n$ is a map defined, for $\phi \in C([-\Delta, 0]; R^n)$, as

$$\mathcal{D}\phi = \phi(0) - q(\phi); \tag{8}$$

f, q Lipschitz on bounded sets.

So the equation (7) is read as follows

$$\frac{d}{dt}(x(t) - q(x_t)) = f(x_t, v(t)), \qquad t \geq 0,\ a.e. \tag{9}$$

F$\mathcal{D}$Es

Hale, Martinez-Amores, Kolmanovskii, Myshkis, Verriest, Rasvan, Niculescu, Fridman, Gu, Melchor-Aguilar, Pepe, Karafyllis, Jiang

A time invariant F$\mathcal{D}$E is an equation of the type

$$\begin{aligned} & x(t) = g(x_t, u(t)), \qquad t \geq 0, \\ & x(\tau) = x_0(\tau),\ \tau \in [-\Delta, 0], \qquad x_0 \in C([-\Delta, 0]; R^n), \\ & u \ continuous, \\ & g \ Lipschitz \ on \ bounded \ sets, \ independent \ of \ the \ first \ argument \ at \ 0. \end{aligned} \tag{10}$$

Definition 9. *(see Hale & Lunel, 1993) A map* $g : C([-\Delta, 0]; R^n) \times R^m \to R^n$ *is said to be independent of the first argument at 0 if there exists a real* $c \in (0, \Delta]$ *such that, for any* $v \in R^m$ *and for any* $\phi_1,\ \phi_2 \in C([-\Delta, 0]; R^n)$ *satisfying* $\phi_1(\tau) = \phi_2(\tau), \tau \in [-\Delta, -c]$, *the equality* $g(\phi_1, v) = g(\phi_2, v)$ *holds.*

By the independence assumption, the F$\mathcal{D}$E is not implicit, the solution exists in R^+. Assuming the matching condition ($x_0(0) = g(x_0, u(0))$) (naturally satisfied by difference maps involved in NFDEs in Hale's form), the solution is continuous.

Definition 10. *Let in the $F\mathcal{D}E$ (10) $u(t) = 0\ \forall t \geq 0$. The system described by the F$\mathcal{D}$E (10) is said to be* $0-$*GAS if there exists a function β of class $\mathcal{KL}$ such that, for any $x_0 \in C([-\Delta, 0]; R^n)$, the corresponding solution satisfies the inequality*

$$|x(t)| \leq \beta(\|x_0\|_\infty, t),\ \forall t \geq 0 \tag{11}$$

Definition 11. *The system described by the F$\mathcal{D}$E (10) is said to be ISS, if there exist a function β of class $\mathcal{KL}$ and a function γ of class $\mathcal{K}$ such that, for any $x_0 \in C([-\Delta, 0]; R^n)$ and any continuous input signal u, satisfying the matching condition, the corresponding solution satisfies*

$$|x(t)| \leq \beta(\|x_0\|_\infty, t) + \gamma(\|u_{[0,t]}\|_\infty), \qquad \forall t \geq 0$$

Let, for any continuous function $w : [0, c] \to R^m$ and any $\phi \in C([-\Delta, 0]; R^n)$, satisfying the matching condition $\phi(0) = g(\phi, w(0))$, $\phi_{c,w} \in C([-\Delta, 0]; R^n)$ be defined, for $s \in [-\Delta, 0]$, as

$$\phi_{c,w}(s) = \begin{cases} \phi(s+c), & s \in [-\Delta, -c) \\ g(\phi_s^\star, w(s+c)), & s \in [-c, 0], \end{cases} \tag{12}$$

where $\phi_s^\star \in C([-\Delta, 0]; R^n)$ is defined, for $\theta \in [-\Delta, 0]$, $s \in [-c, 0]$, as

$$\phi_s^\star(\theta) = \begin{cases} \phi(\theta + s + c), & \theta \in [-\Delta, -c-s) \\ \phi(0), & \theta \in [-c-s, 0], \end{cases} \tag{13}$$

Theorem 12. *(Pepe, AUT, 2014) The system described by the FDE (10), with $u(t) = 0\ \forall t \geq 0$, is $0-$GAS if and only if there exists a continuous functional $V : C([-\Delta, 0]; R^n) \to R^+$, functions α_1, α_2 of class $\mathcal{K}_\infty$, a function α_3 of class $\mathcal{K}$, a semi-norm $\|\cdot\|_a$ in $C([-\Delta, 0]; R^n)$ such that, $\forall \phi \in C([-\Delta, 0]; R^n) : \phi(0) = g(\phi, 0)$, the inequalities hold:*

i) $\alpha_1(|\phi(0)|) \leq V(\phi) \leq \alpha_2(\|\phi\|_a)$;

ii) $V(\phi_{c,0}) - V(\phi) \leq -\alpha_3(\|\phi\|_a)$

Recall that $\underline{\gamma}_a|\phi(0)| \leq \|\phi\|_a \leq \overline{\gamma}_a\|\phi\|_\infty$.

Theorem 13. *(Pepe, AUT, 2014) The system described by the $\mathcal{FDE}$ (10) is ISS if and only if there exists a continuous functional*

$$V : C([-\Delta, 0]; R^n) \to R^+,$$

functions $\alpha_1, \alpha_2, \alpha_3$ of class $\mathcal{K}_\infty$, a function σ of class $\mathcal{K}$, a semi-norm $\|\cdot\|_a$ in $C([-\Delta, 0]; R^n)$ such that, for any $\phi \in C([-\Delta, 0]; R^n)$ and any continuous function $w : [0, c] \to R^m$, satisfying the matching condition $\phi(0) = g(\phi, w(0))$, the inequalities hold:

i) $\alpha_1(|\phi(0)|) \leq V(\phi) \leq \alpha_2(\|\phi\|_a)$;

ii) $V(\phi_{c,w}) - V(\phi) \leq -\alpha_3(\|\phi\|_a) + \sigma(\sup_{\tau \in [0,c]} |w(\tau)|)$

Recall that $\underline{\gamma}_a |\phi(0)| \leq \|\phi\|_a \leq \overline{\gamma}_a \|\phi\|_\infty$.

$$\frac{d}{dt}(\mathcal{D}x_t) = f(x_t, v(t)), \qquad t \geq 0,\ a.e.,$$
$$x(\tau) = x_0(\tau),\ \tau \in [-\Delta, 0],\ x_0 \in C([-\Delta, 0]; R^n), \tag{14}$$

where: $x(t) \in R^n$; $v(t) \in R^m$ is the input, measurable and locally essentially bounded, n, m are positive integers; $\mathcal{D} : C([-\Delta, 0]; R^n) \to R^n$ is a map defined, for $\phi \in C([-\Delta, 0]; R^n)$, as

$$\mathcal{D}\phi = \phi(0) - q(\phi) \qquad (for\ x_t,\ \mathcal{D}x_t = x(t) - q(x_t)); \tag{15}$$

q, f Lipschitz on bounded sets, q independent of the first argument at 0.

Lemma 14. *The following results hold:*

1) *there exist, unique, a continuous solution* $x(t)$ *of the NFDE in Hale's form (14), on a maximal time interval* $[0, b)$, $0 < b \leq +\infty$;

2) *the function* $t \to x(t) - q(x_t)$ *is locally absolutely continuous in* $[0, b)$;

3) *if* $b < +\infty$, *then the function* $t \to x(t) - q(x_t)$, $t \in [0, b)$, *is unbounded in* $[0, b)$.

Definition 15. *(see Hale & Lunel, 1993, Kolmanovskii & Myshkis, 1999, Khalil, 2000) The system described by the NFDE (14), with* $u(t) \equiv 0$*, is said to be* 0*-GAS if there exists a function* β *of class* $\mathcal{KL}$ *such that, for any initial state* $\psi \in C([-\Delta, 0]; R^n)$*, the solution exists for all* $t \geq 0$ *and, furthermore, it satisfies*

$$|x(t)| \leq \beta \left(\|\psi\|_\infty, t \right) \tag{16}$$

Definition 16. *(Sontag, 1989, Pepe, AUT, 2007) The system described by the NFDE (14) is said to be input-to-state stable if there exist a function* β *of class* $\mathcal{KL}$ *and a function* γ *of class* $\mathcal{K}$ *such that, for any initial state* $\psi \in C([-\Delta, 0]; R^n)$ *and any measurable, locally essentially bounded input* v*, the solution exists for all* $t \geq 0$ *and, furthermore, it satisfies*

$$|x(t)| \leq \beta \left(\|\psi\|_\infty, t \right) + \gamma \left(\|v_{[0,t)}\|_\infty \right) \tag{17}$$

For a locally Lipschitz functional $V : C([-\Delta, 0]; R^n) \to R^+$, the derivative of the functional V, $D^+V : C([-\Delta, 0]; R^n) \times R^m \to R^\star$, is defined for $\phi \in C([-\Delta, 0]; R^n)$, $v \in R^m$, as

$$D^+V(\phi, v) = \limsup_{h \to 0^+} \frac{1}{h}\Big(V(\phi_{h,v}) - V(\phi)\Big), \tag{18}$$

where: for $0 < h < \Delta$, $\phi_{h,v} \in C([-\Delta, 0]; R^n)$ is given by

$$\phi_{h,v}(s) = \begin{cases} \phi(s+h), & s \in [-\Delta, -h]; \\ \mathcal{D}\phi + f(\phi, v)(s+h) - \mathcal{D}\phi^\star_{s+h} + \phi(0), \\ & s \in (-h, 0]; \end{cases} \tag{19}$$

for $0 < \theta \leq h$, $\phi^\star_\theta \in C([-\Delta, 0]; R^n)$ is given by

$$\phi^\star_\theta(s) = \begin{cases} \phi(s+\theta), & s \in [-\Delta, -\theta]; \\ \phi(0), & s \in (-\theta, 0] \end{cases} \tag{20}$$

Theorem 17. *(Pepe & Karafyllis, IJC, 2013) Consider the NFDE (14), with the input signal* $v \equiv 0$. *Let there exist a positive integer* p, p *reals* $\Delta_i \in (0, \Delta]$, $i = 1, 2, \ldots, p$ *and* p *matrices* $A_i \in R^{n\times n}$, $i = 1, 2, \ldots, p$ *such that*

$$\mathcal{D}\phi = \phi(0) - \sum_{k=1}^{p} A_k \phi(-\Delta_k) \tag{21}$$

Let the system described by the F$\mathcal{D}$*E*

$$\begin{aligned} &\mathcal{D}x_t = 0, \qquad t \geq 0, \\ &x(\tau) = x_0(\tau), \qquad \tau \in [-\Delta, 0], \qquad x_0 \in C([-\Delta, 0]; R^n), \end{aligned} \tag{22}$$

be strongly stable (see Hale & Lunel, 1993). Then, the system described by the NFDE (14) is 0*-GAS if and only if there exist a locally Lipschitz functional* $V : C([-\Delta, 0]; R^n) \to R^+$*, functions* α_1, α_2 *of class* $\mathcal{K}_\infty$*, and a function* α_3 *of class* $\mathcal{K}$ *such that,* $\forall\ \phi \in C([-\Delta, 0]; R^n)$*:*

$$\begin{aligned} &H_1) \qquad \alpha_1(|\mathcal{D}\phi|) \leq V(\phi) \leq \alpha_2(\|\phi\|_\infty); \\ &H_2) \qquad D^+V(\phi, 0) \leq -\alpha_3(|\mathcal{D}\phi|) \end{aligned} \tag{23}$$

Theorem 18. *(Pepe, Karafyllis & Jiang, SCL, 2017) Consider the NFDE (14). Let the system described by the FDE*

$$\mathcal{D}x_t = v(t), \qquad t \geq 0,$$
$$x(\tau) = x_0(\tau), \qquad \tau \in [-\Delta, 0], \qquad x_0 \in C([-\Delta, 0]; R^n), \tag{24}$$

be ISS with respect to the continuous input signal $v(t)$. Then, the system described by the NFDE (14) is ISS if and only if there exist a locally Lipschitz functional $V : C([-\Delta, 0]; R^n) \to R^+$, functions α_1, α_2, α_3 and $\overline{\gamma}_a$ of class $\mathcal{K}_\infty$, a functional $N_a : C([-\Delta, 0]; R^n) \to R^+$, and a function ρ of class $\mathcal{K}$ such that:

$H_1)$ $\alpha_1(|\mathcal{D}\phi|) \leq V(\phi) \leq \alpha_2(N_a(\phi)), \quad \forall\ \phi \in C([-\Delta, 0]; R^n);$

$H_2)$ $N_a(\phi) \leq \overline{\gamma}_a(\|\phi\|_\infty), \quad \forall\ \phi \in C([-\Delta, 0]; R^n);$

$H_3)$ $D^+V(\phi, u) \leq -\alpha_3(N_a(\phi)) + \rho(|u|), \ \forall\ \phi \in C([-\Delta, 0]; R^n),\ u \in R^m$

Retarded, Control-Affine, Nonlinear Systems

$$\begin{aligned} \dot{x}(t) &= f(x_t) + g(x_t)v(t), \qquad t \geq 0, \qquad a.e., \\ x(\tau) &= \xi_0(\tau), \qquad \tau \in [-\Delta, 0], \end{aligned} \tag{25}$$

$$x_t \in C([-\Delta, 0]; R^n), \qquad x_t(\tau) = x(t+\tau)$$

ISS-ation w.r.t. the Actuator Disturbance

$$\dot{x}(t) = f(x_t) + g(x_t)(u(t) + d(t)), \qquad t \geq 0, \qquad a.e.,$$
$$x(\tau) = \xi_0(\tau), \qquad \tau \in [-\Delta, 0], \tag{26}$$

$x(t) \in R^n$, $u(t) \in R^m$ control input, $d(t) \in R^m$ unknown disturbance, supposed Lebesgue measurable and locally essentially bounded.

PROBLEM: given a state feedback $k(x_t)$ such that $\dot{x}(t) = f(x_t) + g(x_t)k(x_t)$ is 0-GAS, find a new state feedback $k(x_t)+p(x_t)$ such that $\dot{x}(t) = f(x_t) + g(x_t)(k(x_t)+p(x_t)+d(t))$ is ISS w.r.t. $d(t)$.

For given $\phi \in C([-\Delta, 0]; R^n)$, $h \in [0, \Delta)$, let $\phi_h^g \in C([-\Delta, 0]; R^{n\times m})$ be defined as

$$\phi_h^g(s) = \begin{cases} 0 & s \in [-\Delta, -h) \\ (s+h)g(\phi) & s \in [-h, 0] \end{cases} \tag{27}$$

Theorem for ISS-ation

Pepe, TAC, 2009

Hp) There exist a Lipschitz on bounded sets functional

$$k : C([-\Delta, 0]; R^n) \to R^m,$$

a continuously Fréchet differentiable functional

$$V : C([-\Delta, 0]; R^n) \to R^+,$$

functions α_1, α_2 and α_3 of class $\mathcal{K}_\infty$, such that, along the solutions of the unforced (disturbance equal to zero) closed loop system (26) with $u(t) = k(x_t)$, described by

$$\dot{x}(t) = f(x_t) + g(x_t)k(x_t), \qquad (28)$$

the following inequalities hold:

i) $\alpha_1(|\phi(0)|) \leq V(\phi) \leq \alpha_2(\|\phi\|_a)$; ii) $D^+V(\phi) \leq -\alpha_3(\|\phi\|_a)$

Th) The feedback control law

$$u(t) = k(x_t) + p(x_t), \tag{29}$$

with $p = \begin{bmatrix} p_1 & p_2 & \dots & p_m \end{bmatrix}^T : C([-\Delta, 0]; R^n) \to R^m$ defined as

$$p_i(\phi) = -\limsup_{h \to 0^+} D_F V(\phi) \frac{1}{h} \phi_h^g e_i, \tag{30}$$

e_i being the canonical basis in R^m, is such that the closed loop system (26), (29), described by

$$\dot{x}(t) = f(x_t) + g(x_t)k(x_t) + g(x_t)p(x_t) + g(x_t)d(t), \tag{31}$$

is input-to-state stable with respect to the measurable and locally essentially bounded disturbance $d(t)$, provided that the functional p is Lipschitz on bounded sets of $C([-\Delta, 0]; R^n)$.

briefly...

$$\dot{x}(t) = f(x_t) + g(x_t)(u(t) + d(t)), \tag{32}$$

Hp) $u(t) = k(x_t)$ is stabilizing in the unforced case ($d(t) = 0$), V is a L-K functional for $\dot{x}(t) = f(x_t) + g(x_t)k(x_t)$

Th) For $p_i(\phi) = \ - \ \limsup_{h \to 0+} \ D_F V(\phi) \ \frac{1}{h} \ \phi_h^g \ e_i$,

$$u(t) = k(x_t) + p(x_t)$$

is input-to-state stabilizing, i.e

$$\dot{x}(t) = f(x_t) + g(x_t)(k(x_t) + p(x_t) + d(t))$$

is ISS w.r.t. $d(t)$.

Disturbance Attenuation

$$|x(t)| \leq \beta(\|\xi_0\|_\infty, t) + \gamma(\|d_{[0,t)}\|_\infty)$$

$$\gamma(s) = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}\left(\frac{s^2}{3}\right)$$

If, instead of V, we choose ωV, with ω a positive real, then

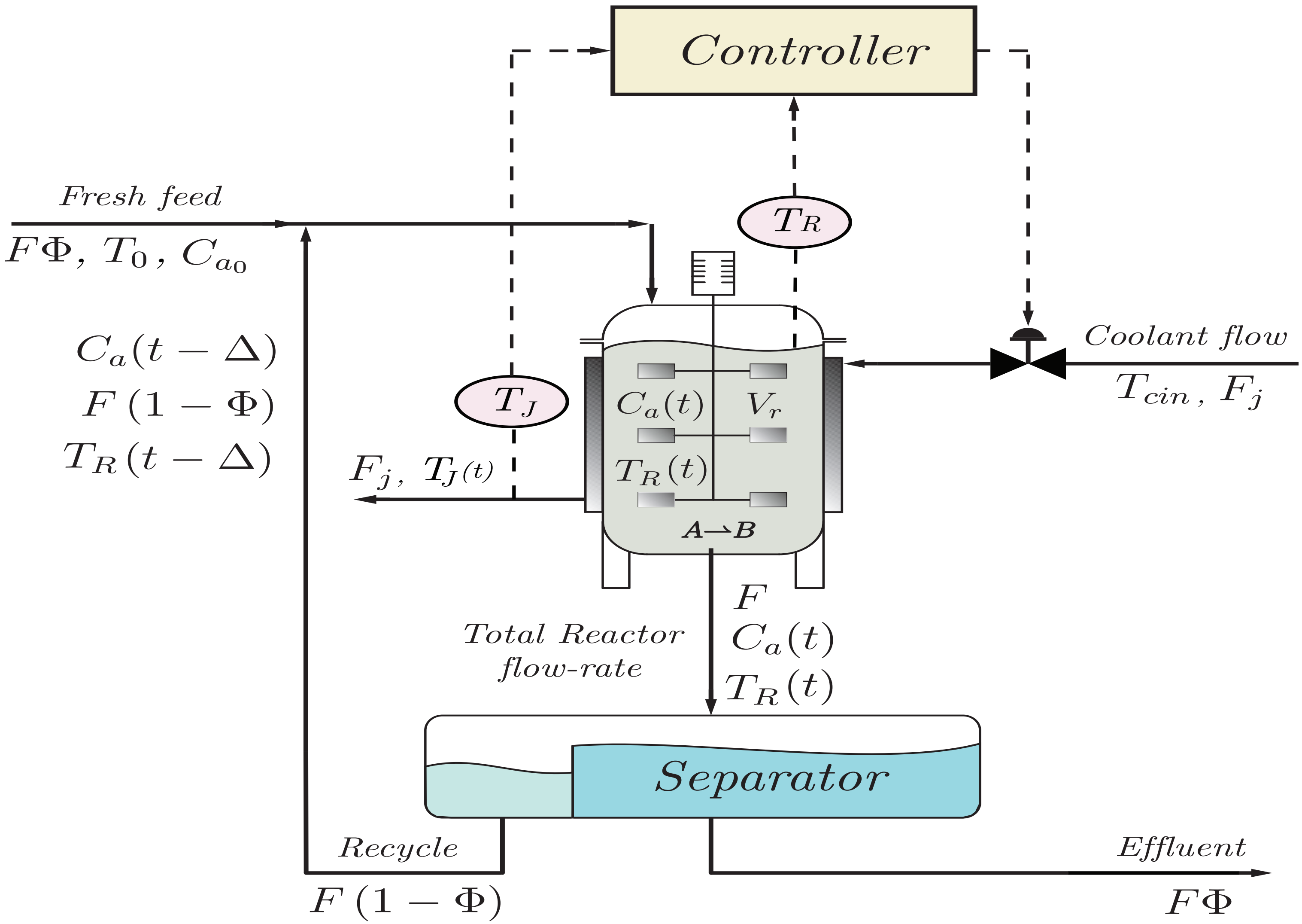
$$\gamma(s) = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}\left(\frac{s^2}{3\ \omega}\right)$$

The disturbance can be arbitrarily attenuated. Price to pay: $p(x_t)$ becomes $\omega p(x_t)$ (i.e., increased control effort).

The case with saturated input and equations with discontinuous right-hand side is investigated in Pepe & Ito, TAC 2012. Invariantly differentiable functionals (Kim, 1997) are used.

Small-gain theory for ISS and integral ISS (iISS) of interconnected systems with delays can be found in

- Karafyllis & Jiang, SIAM, 2007
- Ito, Pepe & Jiang, AUT, 2010
- Ito, Jiang & Pepe, AUT, 2012
- Dashkovskiy, Kosmykov, Mironchenko & Naujok, NAHS, 2012



Scheme of a Continuous Stirred Tank Reactor

CSTR Model, Luyben 2007, Wu, 1999

$$
\begin{aligned}
\frac{dC_A(t)}{dt} &= \frac{F}{V_R}(\phi C_{A0} + (1-\phi)C_A(t-\Delta) - C_A(t)) \\
&\quad - C_A(t)k_0 e^{\frac{-E}{RT_R(t)}} \\
\frac{dT_R(t)}{dt} &= \frac{F}{V_R}(\phi T_0 + (1-\phi)T_R(t-\Delta) - T_R(t)) \\
&\quad - \frac{\lambda C_A(t)k_0 e^{\frac{-E}{RT_R(t)}}}{\rho c_p} - \frac{UA_J(T_R(t)-T_J(t))}{V_R\rho c_p} \\
\frac{dT_J(t)}{dt} &= \frac{F_J(t)}{V_J}(T_{C,in} - T_J(t)) + \frac{UA_J(T_R(t)-T_J(t))}{V_J\rho_J c_J}
\end{aligned} \tag{33}
$$

$F_J(t) = u(t) + d(t)$

In the case the disturbance is not present ($d(t) \equiv 0$), a stabilizing feedback control law

$$u(t) = k((T_R)_t, (C_A)_t, (T_J)_t)$$

is found by tools of differential geometry for time-delay systems (Germani, Manes, Pepe, Oguchi, Watanabe, Nakamizo, Marquez-Martinez, Moog). The closed-loop system (with $u = k$) becomes

$$\dot{E}(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} N(E_1(t)) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{F}{V_R}(1-\Phi) \end{bmatrix} E(t-\Delta)$$
$$+ \begin{bmatrix} A_B + B_B K & & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 \quad 0 & & -\frac{F}{V_R} - k_0 e^{\frac{-E}{R(E_1(t)+T_{R,eq})}} \end{bmatrix} E(t)$$
$$+ \begin{bmatrix} 0 \\ \frac{UA_J(T_{C,in} - \mathcal{F}(E(t),\ E(t-\Delta)))}{V_J V_R \rho c_p} \\ 0 \end{bmatrix} d(t) \qquad (34)$$

A functional V by which the asymptotic stability of the unforced ($d(t) \equiv 0$) closed loop system can be proved, its Fréchet Differential and the related ISS-ing term p in the control law are the following ($\phi, \psi, \phi_h^g \in C([-\Delta, 0]; R^3)$), (Pepe & Di Ciccio, IJRNC 2011)

$$V(\phi) = \phi^T(0)P\phi(0) + \int_{-\Delta}^{0} \phi^T(\tau)\left(-\frac{\tau}{\Delta}Q_1 + \frac{\tau+\Delta}{\Delta}Q_2\right)\phi(\tau)d\tau$$

$$D_F V(\phi)\psi = 2\phi^T(0)P\psi(0) + 2\int_{-\Delta}^{0} \phi^T(\tau)\left(-\frac{\tau}{\Delta}Q_1 + \frac{\tau+\Delta}{\Delta}Q_2\right)\psi(\tau)d\tau$$

$$D_F V(\phi)\frac{1}{h}\phi_h^g = \frac{1}{h}2\phi^T(0)Phg(\phi) + \frac{1}{h}2\int_{-h}^{0} \phi^T(\tau)\left(-\frac{\tau}{\Delta}Q_1 + \frac{\tau+\Delta}{\Delta}Q_2\right)(\tau+h)g(\phi)d\tau$$

$$p(\phi) = -2\phi^T(0)Pg(\phi)$$

$$
p((T_R)_t,(C_A)_t,(T_J)_t) =
-2\begin{bmatrix} T_R(t)-T_{R,eq} \\ \frac{F}{V_R}(\phi T_0+(1-\phi)T_R(t-\Delta)-T_R(t)) \\ -\frac{\lambda C_A(t)k_0 e^{\frac{-E}{RT_R(t)}}}{\rho c_p}-\frac{UA_J(T_R(t)-T_J(t))}{V_R\rho c_p} \\ C_A(t)-C_{A,eq}\end{bmatrix}^T \cdot
P\begin{bmatrix} 0 \\ \frac{UA_J(T_{C,in}-T_J(t))}{V_J V_R\rho c_p} \\ 0 \end{bmatrix} \tag{35}
$$

$$u(t) = k((T_R)_t, (C_A)_t, (T_J)_t)$$

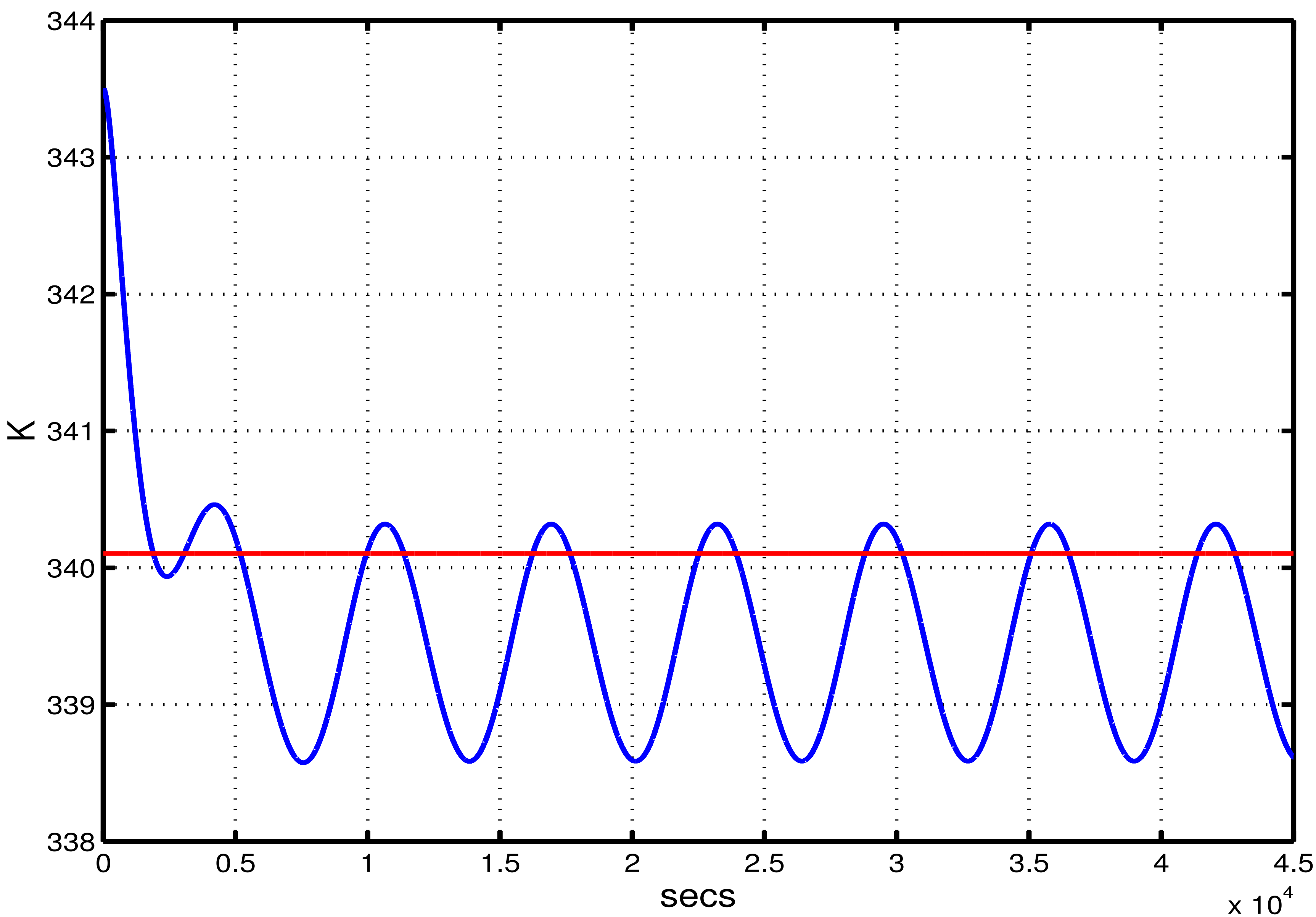
stabilizes (locally) the unforced ($d(t) \equiv 0$) system.

$$u(t) = k((T_R)_t, (C_A)_t, (T_J)_t) + p((T_R)_t, (C_A)_t, (T_J)_t)$$

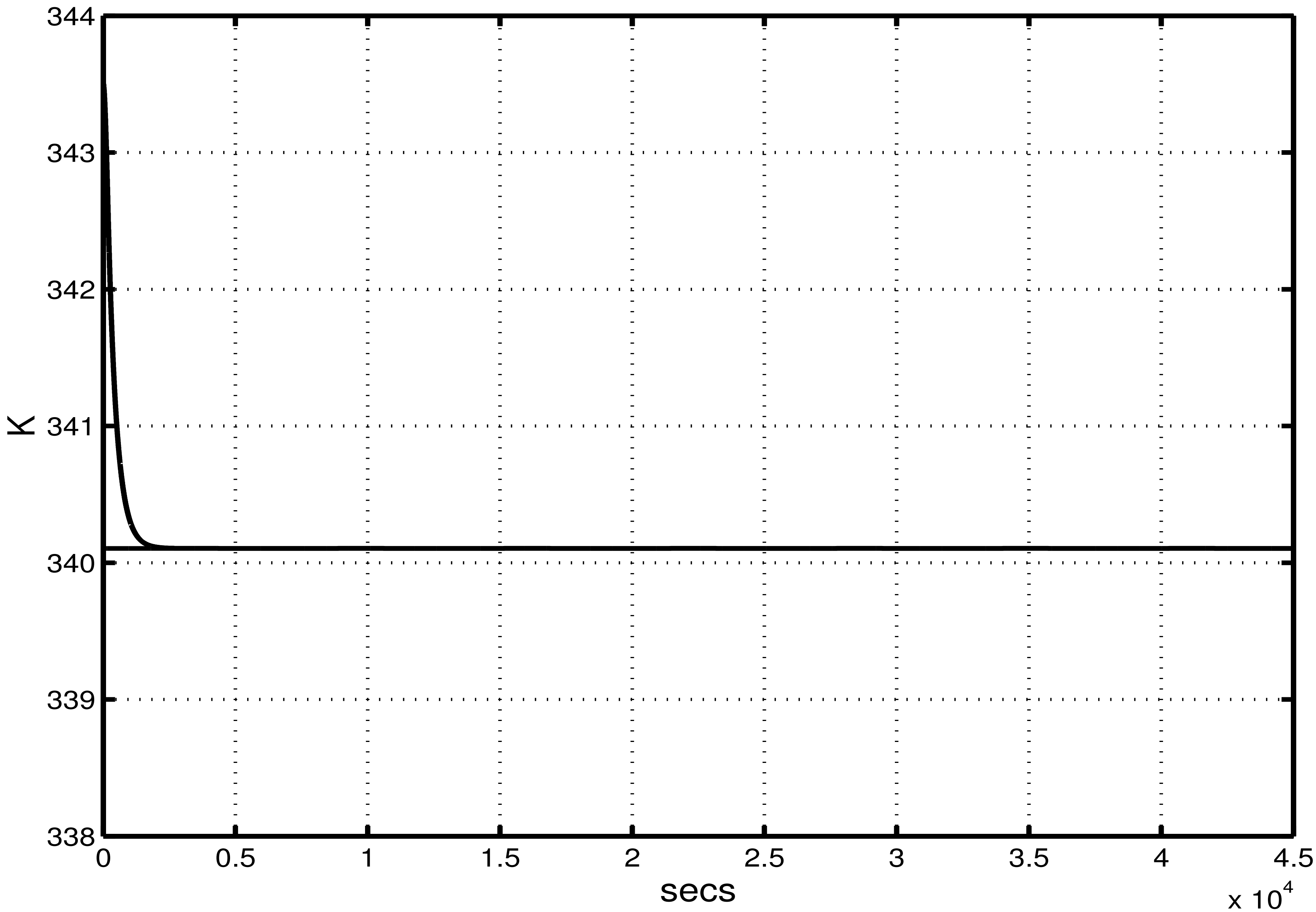
input-to-state stabilizes locally the system with respect to the unknown disturbance $d(t)$ adding to the control input, with significant disturbance effect attenuation.

In the following simulations

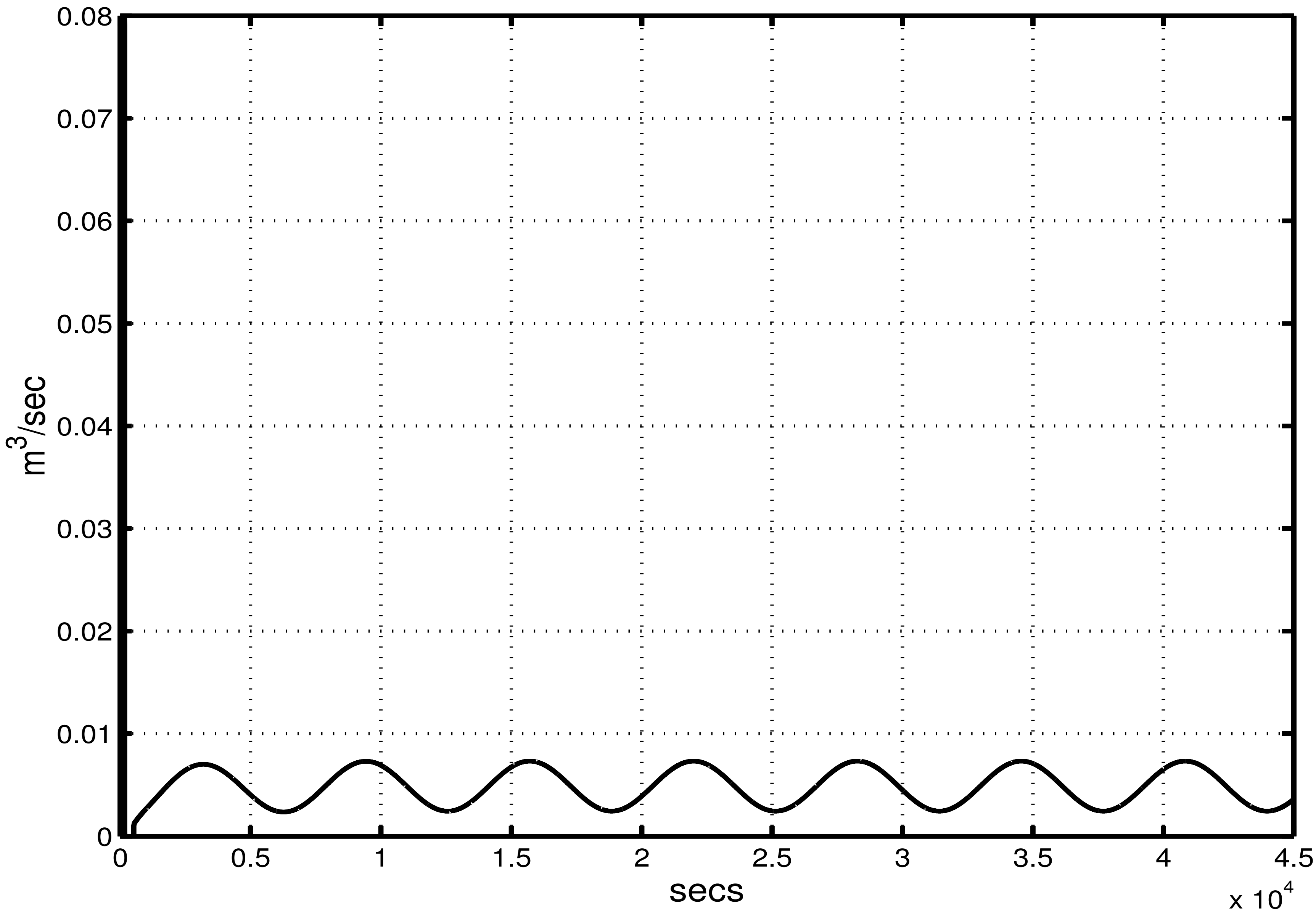
$$d(t) = 0.2F_{J,eq} + 0.4F_{J,eq}cos(0.001t)$$



Reactor Temperature, $u = k$



Reactor Temperature, $u = k + p$



Control Signal, $u = k + p$

Conclusions

- Liapunov-Krasovskii Characterizations of ISS for systems described by RFDEs, F$\mathcal{D}$Es, NFDEs have been presented.

- Formulas for the input-to-state stabilization of retarded nonlinear systems are provided, by means of Fréchet differentiable functionals.

- Such formulas extend the ones given by Sontag in 1989 for delay-free nonlinear systems.

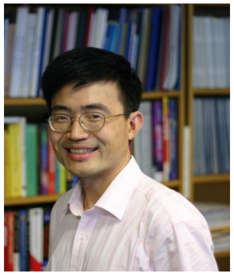
- This theoretical result has been applied to the model of a continuous stirred tank reactor, showing the high performance of the re-designed control law, as far as the attenuation of the actuator disturbance effect is concerned.

Recent Developments

Sontag's formula for the input-to-state stabilization of nonlinear RFDEs and NFDEs has been investigated in Pepe, SCL 2013, 2016.

Robustification of sampled-data stabilizers for RFDEs, by means of ISS redesign with Lyapunov-Krasovskii functionals, has been investigated in Di Ferdinando & Pepe, AUT, 2017.

Thanks to main co-Authors of ISS delay works



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